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Pictures of complete positivity in arbitrary dimension

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ABSTRACT

Two fundamental contributions to categorical quantum mechanics are presented. First, we generalize the CPM-construction, that turns any dagger compact category into one with completely positive maps, to arbitrary dimension. Second, we axiomatize when a given category is the result of this construction.

1. Introduction

Since the start of categorical quantum mechanics [2], dagger compactness has played a key role in most constructions, protocol derivations and theorems. To name two:

- Selinger's CPM-construction, which associates to any dagger compact category of pure states and operations a corresponding dagger compact category of mixed states and operations [27];
- Environment structures, an axiomatic substitute for the CPM-construction which proved to be particularly useful in the derivation of quantum protocols [7,12].

It is well known that assuming compactness imposes finite dimension when exporting these results to the Hilbert space model [16]. This paper introduces variations of each the above two results that rely on dagger structure alone, and in the presence of compactness reduce to the above ones. Hence, these variations accommodate interpretation not just in the dagger compact category of finite dimensional Hilbert spaces and linear maps, but also in the dagger category of Hilbert spaces of arbitrary dimension and continuous linear maps. We show:

- that the generalized CPM-construction indeed corresponds to the usual definitions of infinite-dimensional quantum information theory;
- that the direct correspondence between the CPM-construction and environment structure (up to the so-called doubling axiom) still carries through.

The next two sections each discuss one of our two variations in turn.

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Earlier work The variation of the CPM-construction relying solely on dagger structure was already publicized by one of the authors as a research report [5]. Here we relate that construction to the usual setting of infinite-dimensional quantum information theory, thereby justifying it in terms of the usual model. An earlier version of this construction appeared in conference proceedings [8]. Whereas composition is not always well-defined there, it did have the advantage that the output category of the construction was automatically small if the input was. The construction here is closer to [5], and has the advantage that it is rigorously well-defined, but the disadvantage that the output category might be large. See the discussion after Proposition 4 and Remark 8. Additionally, we generalize the construction further than [8], to braided monoidal categories that are not necessarily symmetric.

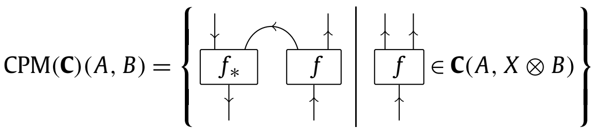
Related work While there are previous results dealing with the transition to a noncompact setting in some way or another, *e.g.* [1,16,17], what is particularly appealing about the results in this paper is that they still allow the diagrammatic representations of braided monoidal categories [21,28].

Future work Classical information can be modeled in categorical quantum mechanics using so-called classical structures [3,10,11]. It is not clear whether these survive CPM-like constructions; see also [18]. The environment structures of Section 3 could be a useful tool in this investigation.

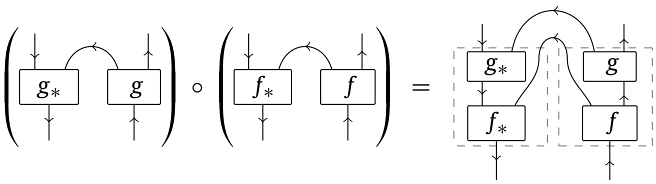
2. Complete positivity

Compact categories and their graphical calculus originated in [23,24]. For a gentle introduction to dagger (compact) categories [2] and their graphical calculus [27], we refer to [9]. We now recall the CPM-*construction* [27], that, given a dagger compact category $\mathbf{C}$, produces a new dagger compact category $\mathrm{CPM}(\mathbf{C})$ as follows. When wires of both types A and A^* arise in one diagram, we will decorate them with arrows in opposite directions. When possible we will suppress coherence isomorphisms in formulae. Finally, recall that $(\_)_*$ reverses the order of tensor products, so f_* has type $A^* \to B^* \otimes C^*$ when $f\colon A \to C \otimes B$ [27].

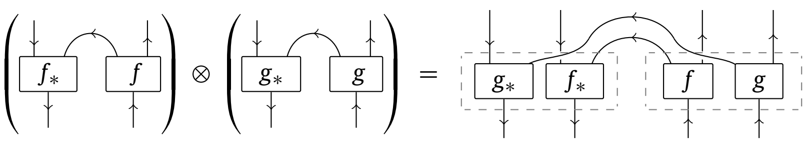
- The objects of $\mathrm{CPM}(\mathbf{C})$ are the same as those of $\mathbf{C}$.
- The morphisms $A \to B$ of $\mathrm{CPM}(\mathbf{C})$ are those morphisms of $\mathbf{C}$ that can be written in the form $(1 \otimes \eta^\dagger \otimes 1)(f_* \otimes f)\colon A^* \otimes A \to B^* \otimes B$ for some morphism $f\colon A \to X \otimes B$ and object X in $\mathbf{C}$.



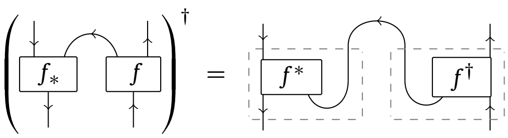
  We call X the *ancillary system* of $(1 \otimes \eta^\dagger \otimes 1)(f_* \otimes f)$, and f its *Kraus morphism*; these representatives are not unique.
- Identities are inherited from $\mathbf{C}$, and composition is defined as follows.



- The tensor unit I and the tensor product of objects are inherited from $\mathbf{C}$, and the tensor product of morphisms is defined as follows.



- The dagger is defined as follows.



- Finally, the cup $\eta_A\colon I \to A^* \otimes A$ in $\mathrm{CPM}(\mathbf{C})$ is given by $(\eta_A)_* \otimes \eta_A = \eta_A \otimes \eta_A$ in $\mathbf{C}$ (*i.e.* with ancillary system I and Kraus morphism η_A in $\mathbf{C}$).