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The expectation monad in quantum foundations

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ABSTRACT

The expectation monad is introduced and related to known monads: it sits between on the one hand the distribution and ultrafilter monad, and on the other hand the continuation monad. The Eilenberg–Moore algebras of the expectation monad are characterized as convex compact Hausdorff spaces, using a theorem of Świrszcz. These convex compact Hausdorff spaces are dually equivalent to Banach (complete) order unit spaces, via a result of Kadison, which in turn are equivalent to Banach effect modules. In this way we obtain a close 'triangle' relationship between predicates and states for the expectation monad. Moreover, the approach leads to a new reformulation of Gleason's theorem, expressing that effects on a Hilbert space are free effect modules on projections, obtained via tensoring with the unit interval.

1. Introduction

Techniques that have been developed over the last decades for the semantics of programming languages and programming logics are gaining wider significance. In this way a new interdisciplinary area has emerged where researchers from mathematics, (theoretical) physics and (theoretical) computer science collaborate, notably on quantum computation and quantum foundations. The article [5] uses the phrase "Rosetta Stone" for the language and concepts of category theory that form an integral part of this common area.

The present article is also part of this new field. It uses results from programming semantics, topology and (convex) analysis, category theory (esp. monads), logic and probability, and quantum foundations. The origin of this article is an illustration of the connections involved. Previously, the authors have worked on effect algebras and effect modules [26,21,22,16] from quantum logic, which are fairly general structures incorporating both logic (Boolean and orthomodular lattices) and probability (the unit interval $[0,1]$ and fuzzy predicates). By reading completely different work, on formal methods in computer security (in particular the thesis [42]), the expectation monad was noticed. The monad is used in [42,8] to give semantics to a probabilistic programming language that helps to formalize (complexity) reduction arguments from security proofs in a theorem prover. In [42] (see also [4,39]) the expectation monad is defined in a somewhat *ad hoc* manner (see Section 9 for details). Soon it was realized that a more systematic definition of this expectation monad could be given via the (dual) adjunction between convex sets and effect modules. Subsequently the two main parts of the present paper emerged.

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(1) The expectation monad turns out to be related to several known monads as described in the following diagram:

$$\begin{array}{ccc} \big(\text{distribution } \mathcal{D}\big) & \overset{\sigma}{\searrow} & \\ & & \big(\text{expectation } \mathcal{E}\big) \longrightarrow \big(\text{continuation } \mathcal{C}\big) \\ \big(\text{ultrafilter } \mathcal{U}\big) & \underset{\tau}{\nearrow} & \end{array} \qquad (1)$$

where the maps σ and τ are defined in Proposition 34 and Lemma 39 respectively, while the morphism to $\mathcal{C}$ is defined in Lemma 30. The continuation monad $\mathcal{C}$ also comes from programming semantics. But here we are more interested in the connection with the distribution and ultrafilter monads $\mathcal{D}$ and $\mathcal{U}$. Since the algebras of the distribution monad are convex sets and the algebras of the ultrafilter monad are compact Hausdorff spaces (a result known as Manes theorem) it follows that the algebras of the expectation monad must be some subcategory of convex compact Hausdorff spaces. This is made precise by a theorem of Świrszcz, describing convex compact Hausdorff spaces as monadic/algebraic over sets, via the monad that sends a set X to the states of the order unit space $\ell^\infty(X)$ of bounded real-valued functions on X. It turns out that the expectation monad is isomorphic to this monad used by Świrszcz. We give a more concrete desciption of the algebras of the monad using basic notions from Choquet theory, notably barycenters of measures.

(2) Kadison duality describes the dual equivalence between convex compact Hausdorff spaces and Banach complete order unit spaces. Here it is shown how these order unit spaces correspond to effect modules. This allows us to give a proper categorical description of the duality between states and effects (predicates) that is fundamental in quantum theory.

These two parts of the paper may be summarized as follows. There are classical results describing the category $\mathcal{EM}(\mathcal{U})$ of Eilenberg–Moore algebras of the ultrafilter monad $\mathcal{U}$ as:

$$\mathcal{EM}(\mathcal{U}) \overset{[\text{Manes}]}{\simeq} \begin{pmatrix}\text{compact}\\ \text{Hausdorff spaces}\end{pmatrix} \overset{[\text{Gelfand}]}{\simeq} \begin{pmatrix}\text{unital}\\ \text{commutative}\\ C^*\text{-algebras}\end{pmatrix}^{\mathrm{op}}$$

Here we give the following “probabilistic” analogues for the expection monad $\mathcal{E}$:

$$\begin{aligned} \mathcal{EM}(\mathcal{E}) \overset{[\text{Świrszcz}]}{\simeq} \begin{pmatrix}\text{convex compact}\\ \text{Hausdorff spaces}\end{pmatrix} &\overset{[\text{Kadison}]}{\simeq} \begin{pmatrix}\text{Banach order}\\ \text{unit spaces}\end{pmatrix}^{\mathrm{op}} \\ &\simeq \begin{pmatrix}\text{Banach}\\ \text{effect modules}\end{pmatrix}^{\mathrm{op}} \end{aligned}$$

The role played by the two-element set $\{0, 1\}$ in these classical results—*e.g.* as “schizophrenic” object—is played in our probabilistic analogues by the unit interval $[0, 1]$.

Quantum mechanics is notoriously non-intuitive. Hence a proper mathematical understanding of the relevant phenomena is important, certainly within the emerging field of quantum computation. It seems fair to say that such an all-encompassing understanding of quantum mechanics does not exist yet. For instance, the categorical analysis in [1,2] describes some of the basic underlying structure in terms of monoidal categories, daggers, and compact closure. However, an integrated view of logic and probability is still missing. Here we certainly do not provide this integrated view, but we indicate a direction in which one might progress towards that goal. The states of a Hilbert space $\mathcal{H}$, described as density matrices $\mathrm{DM}(\mathcal{H})$, fit within the category of convex compact Hausdorff spaces investigated here. Also, the effects $\mathrm{Ef}(\mathcal{H})$ of the space fit in the associated dual category of Banach Hausdorff spaces. The duality we obtain between convex compact Hausdorff spaces and Banach effect algebras precisely captures the translations back and forth between states and effects, as expressed by the isomorphisms:

$$\mathrm{Hom}\big(\mathrm{Ef}(\mathcal{H}), [0, 1]\big) \cong \mathrm{DM}(H) \quad \mathrm{Hom}\big(\mathrm{DM}(\mathcal{H}), [0, 1]\big) \cong \mathrm{Ef}(H).$$

These isomorphisms (implicitly) form the basis for the quantum weakest precondition calculus described in [13].

In this context we shed more light on the relation between quantum logic—as expressed by the projections $\mathrm{Pr}(\mathcal{H})$ on a Hilbert space—and quantum probability—via its effects $\mathrm{Ef}(\mathcal{H})$. In Section 8 it will be shown that Gleason's famous theorem, expressing that states are probability measures, can equivalently be expressed as an isomorphism relating projections and effects:

$$[0, 1] \otimes \mathrm{Pr}(\mathcal{H}) \cong \mathrm{Ef}(\mathcal{H}).$$

This means that the effects form the free effect module on projections, via the free functor $[0, 1] \otimes (-)$. More loosely formulated: quantum probabilities are freely obtained from quantum predicates.

We briefly describe the organization of the paper. It starts with a quick recap on monads in Section 2, including descriptions of the monads relevant in the rest of the paper. Section 3 gives a brief introduction to effect algebras and effect modules. It also establishes equivalences between (Banach) order unit spaces and (Banach) Archimedean effect modules, and the relevant theorems of Świrszcz and Kadison. In Section 4 we give several descriptions of the expectation monad in terms effect modules, states and measures. We also describe the map between the expectation monad and the continuation

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