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Equivariant algorithms for constraint satisfaction problems over coset templates ☆

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ABSTRACT

We investigate the Constraint Satisfaction Problem (CSP) over templates with a group structure, and algorithms solving CSP that are *equivariant*, i.e. invariant under a natural group action induced by a template. Our main result is a method of proving the implication: if CSP over a coset template T is solvable by a local equivariant algorithm then T is 2-Helly (or equivalently, has a majority polymorphism). Therefore bounded width, and definability in fixed-point logics, coincide with 2-Helly. Even if these facts may be derived from already known results, our new proof method has two advantages. First, the proof is short, self-contained, and completely avoids referring to the omitting-types theorems. Second, it brings to light some new connections between CSP theory and descriptive complexity theory, via a construction generalizing CFI graphs.

1. Introduction

Many natural computational problems may be seen as instantiations of a generic framework called *constraint satisfaction problems* (CSP). In a nutshell, a CSP is parametrized by a *template*, a finite relational structure T; the CSP over T asks if a given relational structure I over the same vocabulary as T admits a homomorphism to T (called a solution of I). For every template T, the CSP over T (denoted CSP(T)) is always in NP; a famous conjecture due to Feder and Vardi [14] says that for every template T, the CSP(T) is either solvable in P, or NP-complete.

We concentrate on *coset templates* where, roughly speaking, both the carrier set and the relations have a group structure. The coset templates are cores and admit a Malcev polymorphism, and are thus in P [13,7]. A coset template T naturally induces a group action on (partial) solutions. If, roughly speaking, the induced group action can be extended to the state space of an algorithm solving CSP(T), and the algorithm execution is invariant under the group action, we call the algorithm *equivariant*. We investigate equivariant algorithms which are *local*, i.e. update only a bounded amount of data in every single step of execution.

A widely studied family of local equivariant algorithms is the *local consistency algorithms* that compute families of partial solutions of bounded size conforming to a local consistency condition. Templates T whose CSP(T) is solvable by a local consistency algorithm are said to have *bounded width*. Another source of examples of local equivariant algorithms are logics (via their decision procedures); relevant logics for us will be fix-point extensions of first order logic, like LFP or IFP or IFP+C (IFP with counting quantifiers) [12]. We say that CSP(T) is definable in a logic if some formula of the logic defines the set of all solvable instances of CSP(T).

Our technical contribution is the proof of the following implication: if CSP(T), for a coset template T, is solvable by a local equivariant algorithm then T is *2-Helly*. In consequence, all local equivariant algorithms that can capture 2-Helly templates are equally expressive. The 2-Helly property says that for every partial solution h of an instance I,

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if h does not extend to a solution of I then the restriction of h to some two elements of its domain does not either. This is a robust property of templates with many equivalent characterizations (e.g. strict width 2, or existence of a majority polymorphism) [14]. As a corollary we obtain equivalence of the following conditions for coset templates: (i) 2-Helly; (ii) bounded width; and (iii) definability in fix-point extensions of first-order logic. The corollary is not a new result; equivalence of the first two conditions may be inferred e.g. from Lemma 9 in [11] (even for all core templates with a Malcev polymorphism), while equivalence of the last two ones follows from [1] together with the results of [3] (cf. also [4]). All these results build on Tame Congruence Theory [15], and their proofs are a detour through the deep omitting-type theorems, cf. [16]. Contrarily to this, our proof has an advantage of being short, elementary, and self-contained, thus offering a direct insight into the problem.

Finally, our proof brings to light interesting connections between the CSP theory and the descriptive complexity theory: the crucial step of the proof is essentially based on a construction similar to *CFI graphs*, the intricate construction of Cai, Fürer and Immerman [8]. CFI graphs have been designed to separate properties of relational structures decidable in polynomial time from IFP+C. A similar construction has been used later in [6] to show lack of determination of Turing machines in sets with atoms [5]. The crucial step of our proof is actually a significant generalization of the construction of [6].

For completeness we mention a recent paper of Barto [2] which announces the collapse of bounded width hierarchy for *all* templates: bounded width implies width $(2,3)$, which is however weaker than 2-Helly in general.

2. Preliminaries

2.1. Constraint satisfaction problems

A *template* T is a finite relational structure, i.e. consists of a finite carrier set T (denoted by the same symbol as a template) and a finite family of relations in T. Each relation $R \subseteq T^n$ is of a specified arity, $\text{arity}(R) = n$. Let T be fixed henceforth.

An instance I over a template T consists of a finite set I of elements, and a finite set of *constraints*. A constraint, written $R(a_1, \ldots, a_n)$, is specified by a template relation R and an n-tuple of elements of I, where $\text{arity}(R) = n$.

A partial function h from I to T, with $\{a_1, \ldots, a_n\} \subseteq \text{dom}(h)$, *satisfies* a constraint $R(a_1, \ldots, a_n)$ in I when $R(h(a_1), \ldots, h(a_n))$ holds in T. If h satisfies all constraints in its domain, h is a *partial solution* of I, and h is a *solution* when it is total. By the size of a partial solution h we mean the size of $\text{dom}(h)$. The constraint satisfaction problem over T, denoted $\text{CSP}(T)$, is a decision problem that asks if a given instance over T has a solution.

There are many equivalent formulations of the problem. For instance, one can see I and T as relational structures over the same vocabulary, and then $\text{CSP}(T)$ asks if there is a homomorphism from I to T.

2.2. 2-Helly templates

For an instance I over some template, and $k < j$, a (k, j)-*anomaly* is a partial solution h of I of size j that does not extend to a solution, such that restriction of h to every k-element subset of $\text{dom}(h)$ does extend to a solution. Clearly a (k, j)-anomaly is also (k', j)-anomaly, for $k' < k$.

Definition 2.1. A template T is 2-Helly if no instance of T admits a $(2, j)$-anomaly, for $j > 2$.

In other words: for every partial solution h of size $j > 2$, if the restriction of h to every 2-element subset of its domain extends to a solution then h does extend to a solution too. Analogously one may define k-Helly for arbitrary k, which however will not be needed here.

We conveniently characterize 2-Helly templates as follows.

Lemma 2.2. *A template T is 2-Helly iff no instance of T admits a $(k, k+1)$-anomaly, for $k \geq 2$.*

Proof. For one direction, we observe that a $(k, k+1)$-anomaly is also a $(2, k+1)$-anomaly.

For the other direction, consider an instance with some fixed $(2, j)$-anomaly h, for $j > 2$. For every subset $X \subseteq \text{dom}(h)$, the restriction $h|_X$ either extends to a solution of I, or not. Consider the minimal subset X wrt. inclusion such that $h|_X$ does not extend to a solution of I. For all strict subsets $X' \subseteq X$, $f|_{X'}$ extends to a solution, hence $f|_X$ is a $(k-1, k)$-anomaly, where k is the size of X. Note that $k > 2$. □

2.3. The pp-definable relations

We adopt the convention to mention explicitly the free variables of a formula $\phi(x_1, \ldots, x_n)$. In the specific instances I of $\text{CSP}(T)$ used in our proof it will be convenient to use *pp-definable* relations, i.e. relations definable by an existential first-order formula of the form:

$$\phi(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_{n+m}.\ \psi_1 \wedge \ldots \wedge \psi_l, \tag{1}$$

where every subformula ψ_i is an atomic proposition $R(x_{i_1}, \ldots, x_{i_j})$, for some template relation R. The formula ϕ defines the n-ary relation in T containing the tuples

$$(t_1, \ldots, t_n) \in T^n$$

such that the valuation $x_1 \mapsto t_1, \ldots, x_n \mapsto t_n$ satisfies ϕ. The pp-definable relations are closed under projection and intersection.

In the sequel we feel free to implicitly assume that elements of an instance are totally ordered. The implicit order allows us to treat (partial) solutions as tuples, and allows to state the following useful fact:

Fact 2.3. *Let $X \subseteq I$ be a subset of an instance. The set of partial solutions with domain X that extend to a solution of I, if nonempty, is pp-definable.*