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Kernelization using structural parameters on sparse graph classes ☆

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ABSTRACT

We prove that graph problems with finite integer index have linear kernels on graphs of bounded expansion when parameterized by the size of a modulator to constant-treedepth graphs. For nowhere dense graph classes, our result yields almost-linear kernels. We also argue that such a linear kernelization result with a weaker parameter would fail to include some of the problems covered by our framework. We only require the problems to have FII on graphs of constant treedepth. This allows to prove linear kernels also for problems such as Longest-Path/Cycle, Exact-s,t-Path, Treewidth, and Pathwidth, which do not have FII on general graphs.

1. Introduction

Data preprocessing has always been a part of algorithm design. The last decade has seen steady progress in the area of *kernelization*, an area which deals with the design of polynomial-time preprocessing algorithms. These algorithms compress an input instance of a parameterized problem into an equivalent output instance whose size is bounded by some function of the parameter. Parameterized complexity theory guarantees the existence of such *kernels* for problems that are *fixed-parameter tractable*. Some problems admit stronger kernelization in the sense that the size of the output instance is bounded by a polynomial (or even linear) function of the parameter, the so-called *polynomial (or linear) kernels*.

Of great interest are *algorithmic meta-theorems*, results that focus on problem classes instead of single problems. In the area of graph algorithms, such meta-theorems usually have the following form: all problems with a specific property admit, on a specific graph class, an algorithm of a specific type. We are specifically interested in meta-theorems that concern kernelization, for which a solid groundwork already exists. Before we delve into the history, we need to quickly establish the keystone property that drives all these meta-theorems: the notion of *finite integer index* (FII).

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Roughly speaking, a graph problem has FII if there exists a finite set $\mathcal{S}$ of graphs such that every instance of the problem can be represented by a member of $\mathcal{S}$ alongside an integer "offset". This property is the basis of the *protrusion replacement rule* whereby protrusions (pieces of the input graph satisfying certain requirements) are replaced by members of the set $\mathcal{S}$. Finite integer index is an intrinsic property of the problem itself and is not directly related to whether it can be expressed in a certain logic. In particular, expressibility in the monadic second-order logic of graphs with vertices and edges (MSO_2 and its extension to optimization problems abbreviated as $EMSO_2$) does not imply FII (see [1] for sufficiency conditions for a problem expressible in counting MSO to have FII). As an example of this phenomenon, Hamiltonian Path has FII on general graphs whereas Longest Path does not, although both are $EMSO_2$-expressible.

Now, the first steps toward a kernelization meta-theorem appeared in a paper by Guo and Niedermeier who provided a prescription of how to design linear kernels on planar graphs for graph problems which satisfy a certain distance property [2]. Their work built on the seminal paper by Alber, Fellows, and Niedermeier who showed that Dominating Set has a linear kernel on planar graphs [3]. This was followed by the first true meta-theorem in this area by Bodlaender et al. [1] who showed that graph problems that have FII and satisfy a property called *quasi-coverable*[1] admit linear kernels on bounded genus graphs. Shortly after [1] was published, Fomin et al. [4] proved a meta-theorem for linear kernels on H-minor-free graphs, a graph class that strictly contains graphs of bounded genus. A rough statement of their main result states that any graph problem that has FII, is *contraction bidimensional*, and satisfies a *separation property* has a linear kernel on graphs excluding a fixed graph as minor. This result was, in turn, generalized in [5] to H-topological-minor-free graphs, which strictly contain H-minor-free graphs. Here, the problems are required to have FII and to be *treewidth-bounding*: A graph problem is treewidth-bounding if yes-instances have a vertex set of size linear in the parameter, the deletion of which results in a graph of bounded treewidth. Such a vertex set is called a *modulator to bounded treewidth*. Prototypical problems that satisfy this condition are Feedback Vertex Set and Treewidth t-Vertex Deletion,[2] when parameterized by the solution size.

We see that while these meta-theorems (viewed in chronological order) steadily covered larger graph classes, the set of problems captured in their framework diminished as the other precondition(s) became stricter. Surprisingly, this is not due to said preconditions: It turns out that they can be expressed in a unified manner and are therefore equally restrictive. The combined properties of bidimensionality and separability (used to prove the result on H-minor-free graphs) imply that the problem is treewidth-bounding (cf. Lemma 3.2 and 3.3 in [4]). Quasi-coverability on bounded genus graphs implies the same (cf. Lemma 6.4 in [1]). This demonstrates that all three previous meta-theorems on linear kernels implicitly or explicitly used treewidth-boundedness. Hence the diminishing set of problems can be blamed on the increasingly weaker interaction of the graph classes with the problem parameters, not the (only apparently) stricter precondition on the problems.

This insight motivates a different view on previous meta-theorems: problems that have FII admit linear kernels if parameterized by a *treewidth modulator* in classes excluding a topological minor. In small enough classes (bounded genus, apex-minor-free) the natural parameterization of problems satisfying some basic properties (quasi-coverable, contraction-bidimensionality) coincides with the parameterization by a treewidth-modulator. This change in perspective replaces the natural parameter—whose structural impact diminishes in larger sparse graph classes—by an explicit *structural* parameter which retains the crucial interaction between parameter and graph class. It also gives us, as we will see, the freedom to adapt the parameterization to our needs.

The next well-established level in the sparse-graph hierarchy [6] is formed by the classes of *bounded expansion*. The notion was introduced by Nešetřil and Ossona de Mendez [7] and subsumes graph classes excluding a fixed graph as a topological minor. It turns out that for these classes the serviceable parameterization by a treewidth modulator cannot work if we aim for linear kernels: Any graph class $\mathcal{G}$ can be transformed into a class $\tilde{\mathcal{G}}$ of bounded expansion by replacing every graph $G \in \mathcal{G}$ with $\tilde{G}$, obtained in turn by replacing each edge of G by a path on $|V(G)|$ vertices. For problems like Treewidth t-Vertex Deletion and, in particular, Feedback Vertex Set this operation neither changes the instance membership nor does it increase the parameter. As both the problems do not admit kernels of size $O(k^{2-\epsilon})$ unless coNP $\subseteq$ NP/poly, by a result of Dell and Melkebeek [8], a linear kernelization result on bounded-expansion classes of graphs and under the treewidth-modulator parameterization would have to exclude both these natural problems.

In this work, we identify a structural parameter that indeed does allow linear kernels for all problems that have FII on graph classes of bounded expansion—the size of a tree*depth* modulator. This parameter not only increases under replacing edges with paths (a necessary prerequisite as we now know), but it also provides exactly the structure that seems necessary to obtain such a result. To put this parameterization into context, let us recap some previous work on structural parameters. Even outside the realm of sparse graphs, they have been used to zero in on those aspects of problems that make them intractable—a development that certainly fits the overall agenda of parameterized complexity. This research of alternative parameterizations has given rise to what is called the *parameterized ecology* [9].

Already the perhaps strongest structural parameter for graph-related problems—the vertex cover number—makes up an interesting niche of said ecology, as we summarize now. Many problems that are W-hard or otherwise difficult to parameterize such as Longest Path [10], Cutwidth [11], Bandwidth, Imbalance, Distortion [12], List Coloring, Precoloring Extension, Equitable Coloring, $L(p,1)$-Labeling, and Channel Assignment [13] are (easily) fixed-parameter tractable (*fpt*) when parameterized by the vertex cover number. Some generalizations of vertex cover have also been successfully used as

[1] This property was called *quasi-compactness* in earlier version of [1].
[2] For problem definitions, see Appendix.