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Roland Glantz, Henning Meyerhenke

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On finding convex cuts in general, bipartite and plane graphs[1]

Roland Glantz and Henning Meyerhenke
Institute of Theoretical Informatics
Karlsruhe Institute of Technology (KIT)
Box 6980, D-76128 Karlsruhe, Germany
Email: {roland.glantz, meyerhenke}@kit.edu

Corresponding author: Roland Glantz

Abstract

The general notion of convexity also applies to a graph $G = (V, E)$. A subset V^c of V is called a convex set if all shortest paths with end vertices in V^c remain within V^c. A convex cut of G is a partition (V_1, V_2) of V such that V_1 and V_2 are convex sets (halfspaces). Finding convex cuts is $\mathcal{NP}$-hard for general graphs.

In this paper we first characterize the convex cuts of a connected *bipartite* graph G' in terms of the Djoković relation, a reflexive and symmetric relation on the edges of a graph that is based on shortest paths between the edges' end vertices. Specifically, we show that a cut of G' is convex if and only if the Djoković relation holds for any pair of edges in its cut-set. As a consequence, all convex cuts of $G' = \{V', E'\}$ can be found in $\mathcal{O}(|V'||E'|)$.

We then characterize the convex cuts of a *general* connected graph G using the Djoković-Winkler relation θ and another relation τ on the edges of G. Based on this general characterization, we show how one can find all convex cuts of G in polynomial time. The key parts here are (i) describing the Djoković-Winkler relation on the edges of G in terms of the Djoković relation θ' on the bipartite graph obtained from G by subdividing each edge of G into two edges, (ii) studying the interplay of θ' and τ on plane curves representing convex cuts, and (iii) a running time analysis of our algorithm for finding the convex cuts.

Our method for characterizing and finding convex cuts of a connected plane graph G is motivated by the concept of alternating cuts and conditions on the latter to be convex. In the last part of this paper we represent alternating cuts as plane curves and focus on their intersection pattern. If the plane curves form an arrangement of pseudolines, G is scale embedded in its dual, and any edge of G is contained in the cut-set of a convex cut of G.

Keywords: Convex cuts, Djoković-Winkler relation, bipartite graphs, plane graphs, alternating cuts, pseudolines

1. Introduction

The very general notion of convexity, best known from the field of convex geometry, has an analog on graphs. Let $G = (V, E)$ be a an undirected and unweighted graph. A subset V^c of V is called a *convex set* if all shortest paths on G with end vertices in V^c contain only vertices of V^c. We use the term *convex subgraph* for a subgraph that is induced by a convex set.

A *cut* of a graph $G = (V, E)$ is a partition (V_1, V_2) of V, and the edges between V_1 and V_2 form the *cut-set* of the cut. If V_1 and V_2 are convex sets, (V_1, V_2) is called *convex cut* of G, and V_1, V_2 are called *convex halfspaces*. Artigas et al. [1] showed that, for a given $k \geq 2$, it is $\mathcal{NP}$-complete to decide whether a (general) graph can be partitioned into k convex sets.

Throughout the paper, $d_G(\cdot, \cdot)$ denotes the standard metric on G, i. e., $d_G(u, v)$ amounts to the number of edges on a shortest path from u to v. A graph $G = (V_G, E_G)$ is *isometrically* embeddable [*scale-k embeddable*] into a graph $H = (V_H, E_H)$ if there exists a mapping $\alpha : V_G \mapsto V_H$ such that $d_G(u, v) = d_H(\alpha(u), \alpha(v))$ $[d_H(\alpha(u), \alpha(v)) = k\ d_G(u, v)]$ for all $u, v \in V_G$.

[1]Parts of this paper have been published in a preliminary form in the proceedings of CIAC 2013 [15].