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Complexity and kernels for bipartition into degree-bounded induced graphs

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ABSTRACT

In this paper, we study the parameterized complexity of the problems of partitioning the vertex set of a graph into two parts V_A and V_B such that V_A induces a graph with degree at most a (resp., an a-regular graph) and V_B induces a graph with degree at most b (resp., a b-regular graph). These two problems are called Upper-Degree-Bounded Bipartition and Regular Bipartition, respectively. When $a = b = 0$, the two problems become the polynomially solvable problem of checking the bipartition of a graph. When $a = 0$ and $b = 1$, Regular Bipartition becomes a well-known NP-hard problem, called Dominating Induced Matching. In this paper, firstly we prove that the two problems are NP-complete with any nonnegative integers a and b except $a = b = 0$. Secondly, we show the fixed-parameter tractability of these two problems with parameter $k = |V_A|$ being the size of one part of the bipartition by deriving several problem kernels for them and constrained versions of them.

1. Introduction

In graph algorithms and graph theory, there is a series of important problems which are going to partition the vertex set of a graph into several parts such that each part induces a subgraph satisfying some degree constraints. For example, the k-coloring problem is to partition the graph into k parts each of which induces an independent set (a 0-regular graph). Most of these kinds of problems are NP-hard, even if the problem is to partition a given graph into only two parts, which is called a *bipartition*.

For bipartitions with a degree constraint on each part, we can find many references related to this topic [2,3,7,12,13,16,19]. Here is a definition of the problem:

Degree-Constrained Bipartition
Instance: A graph $G = (V, E)$ and four integers a, a', b and b'.
Question: Is there a partition (V_A, V_B) of V such that

$$a' \leq \deg_{V_A}(v) \leq a \ \ \forall v \in V_A \ \text{ and } \ b' \leq \deg_{V_B}(v) \leq b \ \ \forall v \in V_B,$$

where $\deg_X(v)$ denotes the degree of a vertex v in the induced subgraph $G[X]$?

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There are three special cases of DEGREE-CONSTRAINED BIPARTITION. If there are no constraints on the upper bounds (resp., lower bounds) of the degree in DEGREE-CONSTRAINED BIPARTITION, i.e., $a = b = \infty$ (resp., $a' = b' = 0$), we call the problem LOWER-DEGREE-BOUNDED BIPARTITION (resp., UPPER-DEGREE-BOUNDED BIPARTITION). DEGREE-CONSTRAINED BIPARTITION with the requirement that $a = a'$ and $b = b'$ is called REGULAR BIPARTITION.

In the definitions of the above problems, a, b, a' and b' are part of the input. These problems with some fixed values of a, b, a' and b' have also been studied. We use LOWER-DEGREE-BOUNDED (a', b')-BIPARTITION, UPPER-DEGREE-BOUNDED (a, b)-BIPARTITION and REGULAR (a, b)-BIPARTITION to denote the problems where a and b (or a' and b') are constants.

LOWER-DEGREE-BOUNDED (a', b')-BIPARTITION has been well studied in the literature. LOWER-DEGREE-BOUNDED $(3, 3)$-BIPARTITION with 4-regular graphs is NP-complete [6], while LOWER-DEGREE-BOUNDED $(2, 2)$-BIPARTITION is linear-time solvable [4]. More polynomial-time solvable cases with restrictions on the structure of given graphs and constraints on a' and b' can be found in [2,3,7,12,16].

REGULAR $(0, 0)$-BIPARTITION is the polynomial-solvable problem of checking whether a given graph is bipartite or not; REGULAR $(0, 1)$-BIPARTITION is DOMINATING INDUCED MATCHING, a well studied NP-hard problem also known as EFFICIENT EDGE DOMINATION [10,13,19]. However, not many results are known about UPPER-DEGREE-BOUNDED (a, b)-BIPARTITION and REGULAR (a, b)-BIPARTITION with other constants of a and b.

In this paper, first we show that UPPER-DEGREE-BOUNDED (a, b)-BIPARTITION and REGULAR (a, b)-BIPARTITION are NP-complete with any nonnegative integers a and b except $a = b = 0$. Next, we give several vertex kernels for UPPER-DEGREE-BOUNDED BIPARTITION and REGULAR BIPARTITION by taking the size $k = |V_A|$ of one part as a parameter, where a and b are part of the input: UPPER-DEGREE-BOUNDED BIPARTITION has a kernel of size $O((b+1)^2(b+k)k)$, and REGULAR BIPARTITION has a kernel of size $O((b+1)(b+k)k^2)$ for $a \le b$ or of size $O(k^{k+2})$ for $a > b$. We can assume that $a \le k-1$. The kernel results together with this assumption imply that the two problems are fixed-parameter tractable (FPT) with parameter $k = |V_A|$ when $a \ge b$ and fixed-parameter tractable with parameters $k = |V_A|$ and b when $a < b$. We also discuss the fixed-parameter intractability of our problems with parameter only $k = |V_A|$ when $a < b$.

We notice some related problems, in which the degree constraint on one part of the bipartition changes to a constraint on the size of the part. d-BOUNDED-DEGREE VERTEX DELETION asks us to delete at most k vertices from a graph to make the remaining graph having maximum vertex degree at most d [9]. MAXIMUM d-REGULAR INDUCED SUBGRAPH asks us to delete at most k vertices from a graph to make the remaining graph a d-regular graph [14,15]. VERTEX COVER is the special case of the two problems with $d = 0$. These two problems can be regarded as such a kind of bipartition problems and have been well studied in parameterized complexity. They are FPT with parameters k and d and W[1]-hard with only parameter k [9,14,15,17]. Let tw denote the treewidth of an input graph. Betzler et al. also proved that UPPER-DEGREE-BOUNDED BIPARTITION is FPT with parameters k and tw and W[2]-hard with only parameter tw [5]. The parameterized complexity of some other related problems, such as MINIMUM REGULAR INDUCED SUBGRAPH is studied in [1]. Our $O((b+1)^2(b+k)k)$ kernel for UPPER-DEGREE-BOUNDED BIPARTITION also implies a quadratic vertex kernel for d-BOUNDED-DEGREE VERTEX DELETION for each fixed d, since d-BOUNDED-DEGREE VERTEX DELETION is equivalent to UPPER-DEGREE-BOUNDED $(k-1, d)$-BIPARTITION. The best known kernel for d-BOUNDED-DEGREE VERTEX DELETION is a linear vertex kernel for each fixed $d \ge 0$ [17]. However, the linear vertex kernel cannot be directly extended to UPPER-DEGREE-BOUNDED (a, b)-BIPARTITION for each a and b.

The remaining parts of the paper are organized as follows: Section 2 introduces our notation. Section 3 proves the NP-hardness of our problems. Section 4 gives the problem kernels, and Section 5 shows the fixed-parameter intractability. Finally, some concluding remarks are given in the last section. A preliminary version of this paper [18] was presented in the 25th international symposium on algorithms and computation (ISAAC 2014).

2. Preliminaries

In this paper, a graph stands for a simple undirected graph. We may simply use v to denote the set $\{v\}$ of a single vertex v. Let $G = (V, E)$ be a graph, and $X \subseteq V$ be a subset of vertices. The subgraph induced by X is denoted by $G[X]$, and $G[V \setminus X]$ is also written as $G \setminus X$. Let $E(X)$ denote the set of edges between X and $V \setminus X$. Let $N(X)$ denote the *neighbors* of X, i.e., the vertices $y \in V \setminus X$ adjacent to a vertex $x \in X$, and denote $N(X) \cup X$ by $N[X]$. The *degree* $\deg(v)$ of a vertex v is defined to be $|N(v)|$. A vertex in X is called an *X-vertex*, and a neighbor $u \in X$ of a vertex v is called an *X-neighbor* of v. The number of X-neighbors of v is denoted by $\deg_X(v)$; i.e., $\deg_X(v) = |N(v) \cap X|$. The vertex set and edge set of a graph H are denoted by $V(H)$ and $E(H)$, respectively. When X is equal to $V(H)$ of some subgraph H of G, we may denote $V(H)$-vertices by H-vertices, $V(H)$-neighbors by H-neighbors, and $\deg_{V(H)}(v)$ by $\deg_H(v)$ for simplicity. For a subset $E' \subseteq E$, let $G - E'$ denote the subgraph obtained from G by deleting edges in E'. For an integer $p \ge 1$, a star with $p+1$ vertices is called a *p-star*. The unique vertex of degree > 1 in a p-star with $p > 1$ is called the *center* of the star, and any vertex in a 1-star is a *center* of the star. We may need to find a maximal set of vertex-disjoint p-stars in a given graph, which can be done in polynomial time by iteratively selecting a vertex of degree $\ge p$ together with arbitrary p neighbors and deleting them from the graph.

For a graph G and two nonnegative integers a and b, a partition of $V(G)$ into V_A and V_B is called *(a, b)-bounded* if $\deg_{V_A}(v) \le a$ for all vertices in $v \in V_A$ and $\deg_{V_B}(v) \le b$ for all vertices in $v \in V_B$. An (a, b)-bounded partition (V_A, V_B) is called *(a, b)-regular* if $\deg_{V_A}(v) = a$ for all vertices in $v \in V_A$ and $\deg_{V_B}(v) = b$ for all vertices in $v \in V_B$. An instance $I = (G, a, b)$ of UPPER-DEGREE-BOUNDED BIPARTITION (resp., REGULAR BIPARTITION) consists of a graph G and two nonnegative integers a and b, and asks us to test whether it admits an (a, b)-bounded partition (resp., (a, b)-regular partition) or not. An