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Parameterized complexity of critical node cuts ☆

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ABSTRACT

We consider the following graph cut problem called Critical Node Cut (CNC): Given a graph G on n vertices, and two positive integers k and x, determine whether G has a set of k vertices whose removal leaves G with at most x connected pairs of vertices. We analyze this problem in the framework of parameterized complexity. That is, we are interested in whether or not this problem is solvable in $f(\kappa) \cdot n^{O(1)}$ time (*i.e.*, whether or not it is *fixed-parameter tractable*), for various natural parameters κ. We consider four such parameters:

- The size k of the required cut.
- The upper bound x on the number of remaining connected pairs.
- The lower bound y on the number of connected pairs to be removed.
- The treewidth w of G.

We determine whether or not CNC is fixed-parameter tractable for each of these parameters. We determine this also for all possible aggregations of these four parameters, apart from $w + k$. Moreover, we also determine whether or not CNC admits a polynomial kernel for all these parameterizations. That is, whether or not there is an algorithm that reduces each instance of CNC in polynomial time to an equivalent instance of size $\kappa^{O(1)}$, where κ is the given parameter.

1. Introduction

In 2013 a polio virus struck Israel. The virus spread in alarming speed, creating a nationwide panic of parents concerned about the well-being of their children. It was obvious to the Israeli health department that vaccinating all Israeli children is not a practical solution in the given time frame. Thus it became clear that some areas of the country should be vaccinated first in order to stop the spread of the virus as quickly as possible. Let us represent a geographic area as a vertex of a graph, and the roads between areas as edges of the graph. In this setting, vaccinating an area corresponds to deleting a certain vertex from the graph. Thus, the objective of stopping the virus from spreading translates to minimizing the number of

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Table 1
Summary of the complexity results for CNC. Results with a gray background are either trivial or follow directly from the fact that CNC with $x = 0$ is equivalent to Vertex Cover.

Parameter				Result	
k	x	y	w	FPT	P-Kernel
✓				NO (Theorem 1)	NO (Theorem 1)
	✓			NO	NO
		✓		YES (Theorem 6)	NO (Theorem 7)
			✓	NO (Theorem 4)	NO (Theorem 7)
✓	✓			YES (Theorem 2)	YES (Theorem 3)
✓		✓		YES (Theorem 6)	NO (Theorem 7)
✓			✓	**?**	NO (Theorem 7)
	✓	✓		YES	YES
	✓		✓	YES (Theorem 5)	NO
		✓	✓	YES (Theorem 6)	NO (Theorem 7)
✓	✓	✓		YES	YES
✓	✓		✓	YES (Theorem 2)	YES (Theorem 3)
✓		✓	✓	YES (Theorem 6)	NO (Theorem 7)
	✓	✓	✓	YES	YES
✓	✓	✓	✓	YES	YES

connected pairs (two vertices which are in the same connected component) in the corresponding graph after applying the vaccination.

This scenario can be modeled by the following graph-theoretic problem called Critical Node Cut (CNC). In this problem, we are given an undirected simple graph G and two integers k and x. The objective is to determine whether there exists a set $C \subseteq V(G)$ of at most k vertices in G, such that the graph $G - C$ which results from removing C from G, contains at most x connected pairs. In this sense, the cut C is considered *critical* since removing it from G leaves few (at most x) connected pairs. For convenience, throughout the paper we will count *ordered* connected pairs; *i.e.*, pairs $(u, v) \in V(G) \times V(G)$, $u \neq v$, where u and v belong to same connected component in $G - C$.

The goal of CNC is thus, roughly speaking, to destroy the connectivity of a given graph as much as possible given a certain budget for deleting vertices. From this point of view, CNC fits nicely to the broad family of *graph-cut problems*. Graph-cut problems have been studied widely and are among the most fundamental problems in algorithmic research. Examples include Min Cut, Max Cut, Multicut, Multiway Cut, Feedback Vertex Set, and Vertex Cover (see *e.g.* [23] for definitions of these problems). The latter is the special case of CNC with $x = 0$. Since Vertex Cover is one of the most important problems in the theory of algorithmic design for NP-hard problems, CNC provides a natural test bed to see which of the techniques from this theory can be extended, and to what extent.

1.1. Previous work and applications

The CNC problem has been studied from various angles. The problem was shown to be NP-complete [3] (although its NP-completeness follows directly from the much earlier NP-completeness result for Vertex Cover). In trees, a weighted version of CNC is NP-complete whereas the unweighted version can be solved in polynomial time [15]. The case of bounded treewidth can be solved using dynamic programming in $O(n^{w+1})$ time, where n is the number of vertices in the graph and w is its treewidth [1]. Local search [3] and simulated annealing [32] were proposed as heuristic algorithms for CNC. Finally, in [33] an approximation algorithm based on randomized rounding was developed.

Due to its generic nature, the CNC problem has been considered in various applications. One example application is the virus vaccination problem discussed above [3]. Other interesting applications include protecting a computer/communication network from corrupted nodes, analyzing anti-terrorism networks [27], measuring centrality in brain networks [25], insulin signaling [31], and protein–protein interaction network analysis [8].

1.2. Our results

From reviewing the literature mentioned above, it is noticeable that an analysis of CNC from the perspective of parameterized complexity [17] is lacking. The purpose of this paper is to remedy this situation. We examine CNC with respect to four natural parameters along with all their possible combined aggregations. The four basic parameters we examine are:

- The size k of the solution (*i.e.*, the critical node cut) C.
- The bound x on the number of connected pairs in the resulting graph $G - C$.
- The number of connected pairs y to be removed from G; if G is connected and has n vertices then $y = n(n-1) - x$.
- The treewidth w of G.

Table 1 summarizes all we know regarding the complexity of CNC with respect to these four parameters and their aggregation. Let us briefly go through some of the trivial results given in the table above. First note that CNC with $x = 0$ is precisely