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Scattered packings of cycles ☆

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ABSTRACT

We consider the problem Scattered Cycles which, given a graph G and two positive integers r and ℓ, asks whether G contains a collection of r cycles that are pairwise at distance at least ℓ. This problem generalizes the problem Disjoint Cycles which corresponds to the case $\ell = 1$. We prove that when parameterized by r, ℓ, and the maximum degree Δ, the problem Scattered Cycles admits a kernel on $24\ell^2\Delta^\ell r\log(8\ell^2\Delta^\ell r)$ vertices. We also provide a $(16\ell^2\Delta^\ell)$-kernel for the case $r = 2$ and a $(148\Delta r\log r)$-kernel for the case $\ell = 1$. Our proofs rely on two simple reduction rules and a careful analysis.

1. Introduction

We consider the problem of deciding if a graph contains a collection of cycles that are pairwise far apart. More precisely, given a graph G and two positive integers r and ℓ, we have to decide if there are at least r cycles in G such that the distance between any two of them is at least ℓ. By *distance* between two subgraphs H, H' of a graph G we mean the minimum number of edges in a path from a vertex of H to a vertex of H' in G. This problem, that we call Scattered Cycles, is a generalization of the well-known Disjoint Cycles problem, which corresponds to $\ell = 1$. It is also related to the Induced Minor problem[1] when $\ell = 2$ in the sense that a graph contains r cycles which are pairwise at distance at least 2 if and only if this graph contains $r \cdot K_3$ as induced minor. Hence, any result about the computational complexity of Scattered Cycles gives information on the complexity of Induced Minor. Lastly, it can be seen as an extension of the problems Independent Set and Scattered Set, which instead of cycles, ask for vertices which are far apart.

It is worth noting that besides the connections to other problems mentioned above, this problem has several features which make its study interesting. The first one is that neither positive instances, nor negative ones are minor-closed classes. Therefore the tools from Graph Minors (in particular the Graph Minor Theorem [11]) do not directly provide complexity results for this problem.

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[1] Given two graphs H (guest) and G (host), the Induced Minor problem asks whether H can be obtained from an induced subgraph of G by contracting edges.

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Table 1
Complexity of Scattered Cycles wrt. various parameterizations.

#par	r	ℓ	Δ	Complexity
0	–	–	–	NP-hard (Corollary 5)
1	par	$=1$	–	• FPT (minor checking); • no polynomial kernel unless NP $\subseteq$ coNP/poly [3].
	par	>1	–	W[1]-hard (Corollary 5)
	–	par	–	paraNP-hard (Corollary 5)
	–	–	par	paraNP-hard (Corollary 5)
2	par	$=1$	par	• $O(\Delta r \log r)$-kernel [7]; • $(148\Delta r \log r)$-kernel (Corollary 1).
	par	>1	par	Open
	$=2$	par	par	$(16\ell^2\Delta^\ell)$-kernel (Theorem 2)
	–	par	par	para-NP-hard (Corollary 5)
	par	par	–	W[1]-hard (Corollary 5)
3	par	par	par	$(24\ell^2\Delta^\ell r \log(8\ell^2\Delta^\ell r))$-kernel (Theorem 1)

Also, Scattered Cycles seems unlikely to be expressible in terms of the usual containment relations on graphs. As pointed out above, the cases $\ell = 1$ and $\ell = 2$ correspond to checking if the graph contains $r \cdot K_3$ as minor or induced minor, respectively. However for $\ell > 2$, none of the common containment relations conveys the restriction that cycles have to be at distance at least ℓ. Again, the techniques related to the minor relation cannot be applied immediately.

Lastly, the special case $\ell = 2$ and $r = 2$ corresponds to a question of [4] (also raised in [9,2]) about the complexity of checking whether a graph contains two mutually induced cycles (equivalently, two triangles as induced minor).

Our goal in this paper is to investigate the kernelizability of Scattered Cycles under various parameterizations. Table 1 summarizes known results and the ones that we obtained on the parameterized complexity of the problem with respect to various combinations of parameters, among the number of cycles, the minimum distance required between two cycles and the maximum degree of the graph. A parameterized problem is said to be paraNP-hard if it is NP-hard for some fixed value of the parameter. Unless otherwise specified, we will use all along the paper r, ℓ and Δ to denote, respectively, the number of cycles, the minimum distance allowed between two cycles and the maximum degree of the input graph. As the problem is unlikely to have a polynomial kernel when parameterized by any of these parameters taken alone (cf. Table 1), we naturally explore its kernelizability with several parameters. The first column of the table counts the number of parameters taken into account in a given row, and "par" in the second column indicates that the corresponding value is taken as parameter.

Our results are the following.

Theorem 1. *The problem* Scattered Cycles *admits a kernel on* $24\ell^2\Delta^\ell r \log(8\ell^2\Delta^\ell r)$ *vertices when parameterized by* ℓ, r, *and* Δ. *Moreover this kernel can be computed from an* n*-vertex graph in* $O(n\ell)$ *steps.*

As mentioned above, a trivial consequence of Theorem 1 is that the problem of checking if a graph G contains $r \cdot K_3$ as induced minor admits a $O(\Delta^2 r \log(\Delta^2 r))$-kernel when parameterized by r and Δ.

Theorem 2. *The problem* Scattered Cycles *restricted to* $r = 2$ *admits a kernel on* $16\ell^2\Delta^\ell$ *vertices when parameterized by* ℓ *and* Δ. *Furthermore this kernel can be computed from an* n*-vertex graph in* $O(n\ell)$ *steps.*

Theorem 2 gives a partial answer to a question of [4] about the complexity of checking if graph H contains $2 \cdot K_3$ as induced minor.

The problem known as Disjoint Cycles corresponds to Scattered Cycles for $\ell = 1$. The authors of [3] proved that when parameterized by the number r of cycles only, this problem does not have a polynomial kernel unless NP $\subseteq$ coNP/poly.

For every graph G, let us denote by $\Lambda(G)$ the least non-negative integer t such that G does not contain $K_{1,t}$ as induced subgraph. This parameter refines the one of maximum degree in the sense that for every graph G we have $\Delta(G) + 1 \geq \Lambda(G)$ (hence $\{G,\ \Lambda(G) \leq k+1\} \supseteq \{G,\ \Delta(G) \leq k\}$). An $O(\Lambda r \log r)$-kernel has been provided for the problem Disjoint Cycles parameterized by the number r of cycles and Λ in [7, Corollary 2]. Building upon the techniques used to prove Theorem 1 and ideas from the proof of the aforementioned result, we achieved a bound of the same order of magnitude, with a simpler proof and explicit (small) constants.

Theorem 3. *The problem* Disjoint Cycles *admits a kernel on* $148\Lambda r \log r$ *vertices when parameterized by* Λ *and* r. *This kernel can be computed from an* n*-vertex graph in* $O(n^2)$ *steps.*

As $\Delta(G) + 1 \geq \Lambda(G)$ for every graph G, we immediately obtain the following corollary.