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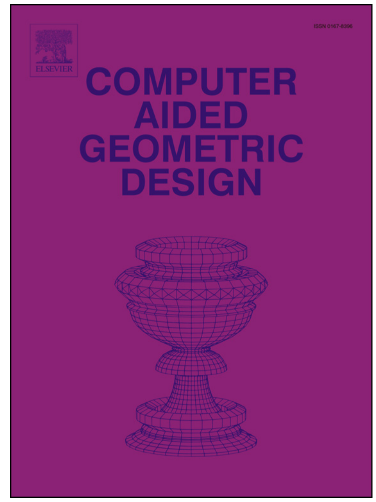


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# Cubic Spline Approximation of a Circle with Maximal Smoothness and Accuracy

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**Abstract**

We construct cubic spline approximations of a circle which are four times continuously differentiable and converge with order six.

Keywords: Bézier curve, cubic spline, geometric smoothness, high accuracy approximation of circles
Mathematics Subject Classification: 41A10, 41A15, 68U05, 68U10, 65Dxx

Applying concepts of differential geometry to CAGD has led to a number of interesting results. An example is the surprising fact that spline curves can approximate with higher order than spline functions. In addition, smoothness constraints become less restrictive. A general statement to this effect is the following

**Conjecture.** For smooth curves in $\mathbb{R}^d$, which satisfy mild generic assumptions, there exist spline approximations of degree $\leq n$ which are $\alpha = n - 1 + \lfloor (n-1)/(d-1) \rfloor$ times continuously differentiable and converge with order $\alpha + 2$.

For low degree, many schemes of optimal order have been proposed (cf. [3, 7] and the references given there for an overview). However, except for the simplest case $n = d = 2$ treated in [6], the gain in accuracy has not yet been combined with a possible improvement of smoothness. This motivated this short note, which, aside from some obvious practical relevance, supports the conjecture for a canonical test case.

The construction of a, in some sense optimal, cubic spline approximation $t \mapsto p(t)$ of a circle is straightforward. We interpolate the circle at the points

$$P_k = (\cos(k\varphi), \sin(k\varphi)), \quad \varphi = 2\pi/m\,,$$

which separate the $m$ cubic Bézier segments of $p$, and choose $k = 0, 1, \ldots, m-1$ as knots. Morerover, we match the tangents at $P_k$. For a completely symmetric configuration, this leaves the length of the tangent vectors as the only degree of freedom. In terms of a corresponding parameter $\delta$, the interior control points of the curve segments are

$$P_k^{\pm} = P_k \pm \delta(-\sin(k\varphi), \cos(k\varphi)), \quad \delta = |p'(k)|/3\,.$$