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Conditions for preserving negative imaginary properties in feedback interconnections and an application to multi-agent systems

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Abstract— **We derived necessary and sufficient conditions for a feedback interconnection of negative imaginary systems to have negative imaginary properties. We first consider two negative imaginary systems (or strictly negative imaginary systems) which are interconnected via positive feedback. Then necessary and sufficient conditions for determining whether the feedback interconnection has negative imaginary properties are presented. These conditions are specialised to a dc loop gain that is easy to check under some assumptions. Together with stability results in the literature, these conditions could be applied to check the stability of a string of several coupled negative imaginary subsystems. Such an example is given to illustrate the usefulness of our proposed results.**

I. INTRODUCTION

An important notion in control analysis and synthesis is the negative imaginary property. The concept of negative imaginary systems was firstly inspired by inertial systems [7]. Systems whose input and output are force and position (or acceleration) often can be modelled as negative imaginary systems. In the single-input, single-output case, the Nyquist plot of a negative imaginary system is below the real axis. Due to the special properties of negative imaginary systems, there have been wide practical applications in various areas, such as large space structures, nano-positioning control [1], [10], [13] and multi-agent networked systems [2], [20], [21]. The notion of negative imaginary systems was first presented in [7] and so was the negative imaginary lemma. This lemma is very useful for stability analysis and control synthesis of negative imaginary systems. Due of its relevance, the work is extended in [23] which considers negative imaginary systems having poles on the imaginary axis excluding the origin. A further extension in [11] allows negative imaginary systems to have free body dynamics.

Beyond the aforementioned development in negative imaginary lemmas and the negative imaginary notion, the theory has been specialised and extended also in several other directions. Here we mention some of them as follows. First, several modifications have generalised and specialised the set of the negative imaginary systems as presented in [7], such as discrete-time systems [4], non-rational systems [3], [5], strongly strict negative imaginary systems [8], infinite dimensional systems [14], Hamiltonian systems [19], descriptor systems [22], finite frequency negative imaginary systems [24], and lossless systems [25]. Second, various control synthesis approaches for negative imaginary systems can be found in [16], [17], [18]. Third, stability results for a positive feedback interconnection of a negative imaginary system and a strictly negative imaginary system are originally proposed in [7] and the results have been modified and applied to different classes of negative imaginary systems shown in [7], [9], [23], [24], [25]. Recently, [6] proposed generalised results that unify all the robust feedback stability work which appeared in earlier literature into one general theory. Fourth, [2] used the work in [7] and discussed the stability for a string of coupled strictly negative imaginary systems.

In this paper, we present necessary and sufficient conditions for negative imaginary properties to be preserved when considering two negative imaginary systems that are interconnected via positive feedback. To achieve this, according to the generalised definition of negative imaginary systems, we propose a negative imaginary lemma without minimality assumptions and this lemma can also be considered as extensions of the work in [11] and [17]. Then, based on the proposed negative imaginary lemma and several auxiliary lemmas, we derive necessary and sufficient conditions for preserving the negative imaginary properties of interconnected negative imaginary systems. The main results together with the stability results given in [6], [7], [23] are used to conclude the stability for a ring of coupled subsystems with negative imaginary dynamics. Unlike the results proposed in [2], which required the systems to be single input, single output, and strictly negative imaginary, our results can be applied to determine the stability for the interconnection of several subsystems without imposing the above restrictions. The main contributions of this work are thus: (i) derivation of a generalised negative imaginary lemma that relaxes the minimality assumptions; (ii) derivation of necessary and sufficient conditions for two strictly negative imaginary systems which are interconnected via positive feedback to remain within the same class of negative imaginary systems; (iii) derivation of generalised results for the positive feedback interconnection of two negative imaginary systems with poles on the imaginary axis except at the origin to preserve negative imaginary properties; (iv) under certain assumptions (i.e. strictly properness of a transfer function), the proposed main results specialise to simple and easy-to-check condition (i.e. dc loop gain condition); and (v) demonstration via a practical example to illustrate the usefulness of the derived

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work.

Notation: $\bar{\lambda}(A)$ is the largest eigenvalue of a square complex matrix that has only real eigenvalues. $\Re(a)$ is a real part of a complex number a. $\mathscr{RH}_\infty$ denotes the set of real, rational, stable transfer function matrices.

II. Preliminary Lemmas

There are different notions of negative imaginary systems. Here, we recall the generalised definition of negative imaginary systems and this definition extends the ones proposed in [7] and [23] to include the case where negative imaginary systems could have possible poles on the imaginary axis and at the origin.

Definition 1: ([9]). Let $R(s)$ be a real, rational, proper transfer function. Then $R(s)$ is said to be Negative Imaginary (NI) if

1) $R(s)$ has no poles in $\Re(s) > 0$;
2) $j[R(j\omega) - R(j\omega)^*] \geq 0$ for all $\omega \in (0, \infty)$ except the values of ω where $j\omega$ is a pole of $R(s)$;
3) if $j\omega_0$ with $\omega_0 > 0$ is a pole of $R(s)$, then it is at most a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0} (s - j\omega_0) jR(s)$ is Hermitian and positive semidefinite;
4) if $s = 0$ is a pole of $R(s)$, then $\lim_{s \to 0} s^k R(s) = 0$ for all integer $k \geq 3$ and $\lim_{s \to 0} s^2 R(s)$ is Hermitian and positive semidefinite.

We also recall the definition of strictly negative imaginary systems as follows.

Definition 2: ([7]). Let $R(s)$ be a real, rational, proper transfer function. Then $R(s)$ is said to be Strictly Negative Imaginary (SNI) if

1) $R(s)$ has no poles in $\Re(s) \geq 0$;
2) $j[R(j\omega) - R(j\omega)^*] > 0$ for all $\omega \in (0, \infty)$.

Unlike negative imaginary systems which could have poles on the imaginary axis, strictly imaginary systems only allow systems to be stable. Besides, in the single-input, single-output case, the Nyquist plot of a negative imaginary system is below the real axis; therefore, the phase of the system must vary between $-\pi$ to 0. Because of Definition 2, the Nyquist plot of strictly negative imaginary systems will never touch the real axis for $\omega \in (0, \infty)$.

Since in this paper we focus on systems which have negative imaginary properties, the negative imaginary lemma is used frequently for deviation of our main results. According to the literature, the first negative imaginary lemma was presented in [7] for stable negative imaginary systems and this result was extended in [23] for the case where negative imaginary systems have poles on the imaginary axis except at the origin; however, minimality assumptions are required in the lemma. Further modifications in [17] relaxed the minimality assumptions and in [11] allowed negative imaginary systems to have possible poles at the origin. Here, we built on these two results and present a generalised version of the negative imaginary lemma which is suitable for the case where negative imaginary systems could have possible poles at the origin without imposing minimality assumptions.

Lemma 3: Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state-space realisation of $R(s)$.

(i) If $D = D^T$ and there exists a $P = P^T \geq 0$ such that

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} \leq 0, \qquad (1)$$

then $R(s)$ is negative imaginary.

(ii) If $R(s)$ is negative imaginary and its state-space realisation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ has no observable uncontrollable modes, then $D = D^T$ and there exists a $P = P^T \geq 0$ satisfying (1).

Proof: The proof will be published elsewhere. ■

III. Interconnected negative imaginary systems via positive feedback

In the literature, there are profound discussions on stability analysis for the positive feedback interconnection of negative imaginary systems. However, if we attempted to consider the stability for several subsystems with negative imaginary properties, which are interconnected via positive feedback, then the work mentioned previously may not be applicable. Reference [2] constrained systems to be single-input, single-output, strictly negative imaginary systems and derived the stability conditions for the coupling of several such subsystems. Unlike [2] which imposed restrictive assumptions, we here derive necessary and sufficient conditions for negative imaginary properties to be preserved when considering two interconnected multi-input, multi-output, negative imaginary systems with possible poles on the imaginary axis via positive feedback and together with the stability results proposed in [23] we establish the stability for a string of coupled negative imaginary subsystems which is illustrated via an example in the next section. Furthermore, if we imposed different and simple assumptions, the necessary and sufficient conditions for a feedback system to be a negative imaginary system presented in this paper simplifies as a dc loop gain condition that is a necessary and sufficient.

A. Two strictly negative imaginary systems

In this subsection, we consider the positive feedback interconnection of two strictly negative imaginary systems as shown in Fig. 1 and derived necessary and sufficient conditions for the feedback system to remain within the class of the negative imaginary systems that satisfies the conditions in Definition 2. The following theorem shows that negative imaginary properties can be carried over when we consider two strictly negative imaginary systems that are interconnected via positive feedback.

Theorem 4: Let $M(s)$ and $N(s)$ be strictly negative imaginary systems. Then the corresponding closed-loop transfer function from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ denoted by $T(s)$ is a proper, strictly negative imaginary system if and only if