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Survey

A still simpler way of introducing interior-point method for linear programming

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ABSTRACT

Linear programming is now included in algorithm undergraduate and postgraduate courses for computer science majors. We give a self-contained treatment of an interior-point method which is particularly tailored to the typical mathematical background of CS students. In particular, only limited knowledge of linear algebra and calculus is assumed.

Contents

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1. Introduction

Terlaky [1] and Lesaja [2] have suggested simple ways to teach interior-point methods. In this paper, we suggest an alternative and maybe still simpler way which is particularly tailored to the typical mathematical background of CS students. In particular, only limited knowledge of linear algebra and calculus is assumed. We have selected most of the material from popular textbooks [3–8] to assemble a self-contained presentation of an interior point method—little of this material is new.

The canonical *linear programming problem* is to

$$\text{minimize } c^T x \text{ subject to } Ax = b \text{ and } x \geq 0. \tag{1}$$

Here, A is an $m \times n$ matrix, c and x are n-dimensional, and b is an m-dimensional vector. A *feasible solution* is any vector x with $Ax = b$ and $x \geq 0$. The problem is *feasible* if there is a feasible solution, and *infeasible* otherwise. A feasible problem is *unbounded* (or more precisely the corresponding objective function is unbounded) if for every real z, there is a feasible x with $c^T x \leq z$, and *bounded* otherwise.

In our presentation, we first assume that feasible solutions to the primal and the corresponding dual LP satisfying a certain set of properties (properties (I1)–(I3) in Section 3) are available. We then show how to iteratively improve these solutions in Sections 2 and 3. In each iteration the gap between the primal and the dual objective value is reduced by a factor $1 - O(1/\sqrt{n})$, where n is the number of variables. The iterative improvement scheme leads to solutions that are arbitrarily close to optimality. In Sections 4 and 5 we discuss how to find the appropriate initial solutions and how to extract an optimal solution from a sufficiently good solution by rounding. Either or both these sections may be skipped in a first course.

Remark 1. It is easy to deal with maximization instead of minimization and with inequality constraints. Indeed, maximize $c^T x$ is equivalent to minimize $-c^T x$. Constraints of type $\alpha_1 x_1 + \cdots + \alpha_n x_n \leq \beta$ can be replaced by $\alpha_1 x_1 + \ldots + \alpha_n x_n + \gamma = \beta$ with a new (slack) variable $\gamma \geq 0$. Similarly, constraints of type $\alpha_1 x_1 + \cdots + \alpha_n x_n \geq \beta$ can be replaced by $\alpha_1 x_1 + \cdots + \alpha_n x_n - \gamma = \beta$ with a (surplus) variable $\gamma \geq 0$.

We consider another problem, the *dual problem*, which is

$$\text{maximize } b^T y, \text{ subject to } A^T y + s = c, \text{ with variables } s \geq 0 \text{ and unconstrained variables } y. \tag{2}$$

The vector y has m components and the vector s has n components. We will call the original problem the *primal problem*.

Claim 1 (*Weak Duality*). *If x is a solution of $Ax = b$ with $x \geq 0$ and (y, s) is a solution of $A^T y + s = c$ with $s \geq 0$, then*

1. *$x^T s = c^T x - b^T y$, and*
2. *$b^T y \leq c^T x$, with equality if and only if $s_i x_i = 0$ for all is.*

Proof. We multiply $s = c - A^T y$ with x^T from the left and obtain

$$\begin{aligned} x^T s &= x^T c - x^T (A^T y) = c^T x - (x^T A^T) y \\ &= c^T x - (Ax)^T y = c^T x - b^T y. \end{aligned}$$

As $x, s \geq 0$, we have $x^T s \geq 0$, and hence, $c^T x \geq b^T y$.

Equality will hold if $x^T s = 0$, or equivalently, $\sum_i s_i x_i = 0$. Since $s_i, x_i \geq 0$, $\sum_i s_i x_i = 0$ if and only if $s_i x_i = 0$ for all i. ■

If x is a feasible solution of the primal and (y, s) is a feasible solution of the dual, the difference $c^T x - b^T y$ is called the *objective value gap* of the solution pair. Thus, if the objective values of a primal feasible and a dual feasible solution are the same, then both solutions are optimal. Actually, from the Strong Duality Theorem, if both primal and dual solutions are optimal, then the equality will hold. We will prove the Strong Duality Theorem in Section 5 (Theorem 2).

If the primal and the dual are both feasible, neither of them can be unbounded as by Claim 1, the objective value of all dual feasible solutions are less than or equal to the objective values of any primal feasible solution. As a consequence: If the primal and the dual are feasible, both are bounded. If the primal is unbounded, the dual is infeasible, and if the dual is unbounded, the primal is infeasible. It may happen that both problems are infeasible. It is also true, that if the primal is feasible and bounded, the dual is feasible and bounded, and vice versa. This is a consequence of strong duality.

We will proceed under the assumption that the primal as well as the dual problem are bounded and feasible. This allows us to concentrate on the core of the interior point method, the iterative improvement scheme. We come back to this point in Section 4.

Claim 1 implies, that if we are able to find a solution to the following system of equations and inequalities

$$Ax = b, \;\; A^T y + s = c, \;\; x_i s_i = 0 \text{ for all } i, \;\; x \geq 0, \;\; s \geq 0,$$

we will get optimal solutions of both the original primal and the dual problem. Notice that the constraints $x_i s_i = 0$ are nonlinear and hence it is not clear whether we have made a step towards the solution of our problem. The idea is now to relax the conditions $x_i s_i = 0$ to the conditions $x_i s_i \approx \mu$ (with the exact form of this equation derived in the next section), where $\mu \geq 0$ is a parameter. We obtain

$$(P_\mu) \quad Ax = b, \;\; A^T y + s = c, \;\; x_i s_i \approx \mu \text{ for all } i, \;\; x > 0, \;\; s > 0.$$

We will show:

1. (initial solution) For a suitable μ, it is easy to find a solution to the problem P_μ. This will be the subject of Section 4.
2. (iterative improvement) Given a solution (x, y, μ) to P_μ, one can find a solution (x', y', s') to $P_{\mu'}$, where μ' is substantially smaller than μ. This will be the subject of Sections 2 and 3. Applying this step repeatedly, we can make μ arbitrarily small.
3. (final rounding) Given a solution (x, y, μ) to P_μ for sufficiently small μ, one can extract an exact solution for the primal and the dual problem. This will be the subject of Section 5.

For the iterative improvement, it is important that $x > 0$ and $s > 0$. For this reason, we replace the constraints $x \geq 0$ and $s \geq 0$ by $x > 0$ and $s > 0$ when defining problem P_μ (see Fig. 1).

Note that $x_i s_i \approx \mu$ for all i implies $b^T y - c^T x \approx n\mu$ by Claim 1. Thus, repeated application of iterative improvement will make the gap between the primal and dual objective values arbitrarily small.

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