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Two modified block-triangular splitting preconditioners for generalized saddle-point problems

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ABSTRACT

For the generalized saddle-point problems, firstly, we introduce a modified generalized relaxed splitting (MGRS) preconditioner to accelerate the convergence rate of the Krylov subspace methods. Based on a block-triangular splitting of the saddle-point matrix, secondly, we propose a modified block-triangular splitting (MBTS). This new preconditioner is easily implemented since it has simple block structure. The spectral properties and the degrees of the minimal polynomials of the preconditioned matrices are discussed, respectively. Moreover, we apply the MGRS and the MBTS preconditioners to three-dimensional linearized Navier–Stokes equations. Then we derive the quasi-optimal parameters of the MGRS and the MBTS preconditioners for two and three-dimensional Navier–Stokes equations, respectively. Finally, numerical experiments are illustrated to show the preconditioning effects of the two new preconditioners.

1. Introduction

In this paper, we consider the generalized saddle-point linear systems arising from the discretization of two-dimensional linearized Navier–Stokes equations with the following special structure [1–3]

$$\mathscr{A}u \equiv \begin{pmatrix} A_1 & O & B_1 \\ O & A_2 & B_2 \\ -B_1^T & -B_2^T & C \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ -g \end{pmatrix} \equiv b, \tag{1.1}$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$ and $A_2 \in \mathbb{R}^{n_2 \times n_2}$ are nonsymmetric positive definite, $B_1 \in \mathbb{R}^{n_1 \times m}$ and $B_2 \in \mathbb{R}^{n_2 \times m}$ are of full column-rank, and $C \in \mathbb{R}^{m \times m}$ is symmetric positive semi-definite. These assumptions guarantee the existence and uniqueness of the solution of the system of linear equations (1.1).

The generalized saddle point-problems arise in computational science and engineering applications, such as computational fluid dynamics, constrained optimization, parameter identification, mixed finite element approximation of second-order elliptic problems or the Stokes equations; see [4–7] and the references therein. In the passed decades, as a class of important iteration algorithms, Krylov subspace methods together with various preconditioners have been used to approximate the solution of saddle-point problems. The frequently used preconditioners of Krylov subspace methods are block diagonal preconditioners [8,9], block triangular preconditioners [10–15], constraint preconditioners [16–20],

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Hermitian and skew-Hermitian splitting (HSS) preconditioners [21–28], positive-definite and skew-Hermitian splitting (PSS) preconditioners [29–32], dimensional splitting (DS) preconditioners [1,2,33] and so on.

For the generalized saddle-point problems (1.1), Cao et al. [34] presented a generalized relaxed splitting (GRS) preconditioner of the form

$$\mathscr{P}_{GRS} = \frac{1}{\alpha}\begin{pmatrix} A_1 & O & O \\ O & \alpha I & O \\ -B_1^T & O & \alpha I \end{pmatrix}\begin{pmatrix} \alpha I & O & B_1 \\ O & A_2 & B_2 \\ O & -B_2^T & \alpha I + C \end{pmatrix}. \tag{1.2}$$

To implement the GRS preconditioner for Krylov subspace methods, we need to solve a system of linear equations at each step of Krylov subspace methods, that is

$$\mathscr{P}_{GRS}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \tag{1.3}$$

Using the following matrix factorization

$$\begin{aligned}\mathscr{P}_{GRS} &= \begin{pmatrix} A_1 & O & O \\ O & I & O \\ -B_1^T & O & I \end{pmatrix}\begin{pmatrix} I & O & O \\ O & I & B_2(\alpha I + C)^{-1} \\ O & O & I \end{pmatrix} \\ &\quad \begin{pmatrix} I & O & O \\ O & A_2 + B_2(\alpha I + C)^{-1}B_2^T & O \\ O & -B_2^T & \alpha I + C \end{pmatrix}\begin{pmatrix} I & O & \frac{1}{\alpha}B_1 \\ O & I & O \\ O & O & I \end{pmatrix},\end{aligned} \tag{1.4}$$

we can solve system of linear equations (1.3) as follows:

Algorithm 1.1. For a given vector $r = [r_1^T, r_2^T, r_3^T]^T$, we can compute the vector $z = [z_1^T, z_2^T, z_3^T]^T$ by (1.4) from the following steps:

(1) solve $A_1 y_1 = r_1$;
(2) solve $(\alpha I + C)y_2 = r_3 + B_1^T y_1$;
(3) solve $(A_2 + B_2(\alpha I + C)^{-1}B_2^T)z_2 = r_2 - B_2 y_2$;
(4) solve $(\alpha I + C)z_3 = r_3 + B_1^T y_1 + B_2^T z$;
(5) $z_1 = y_1 - \frac{1}{\alpha}B_1 z_3$.

Comparing the GRS preconditioner with saddle-point matrix $\mathscr{A}$, we have

$$\mathscr{R}_{GRS} = \mathscr{P}_{GRS} - \mathscr{A} = \begin{pmatrix} O & O & \frac{1}{\alpha}A_1B_1 - B_1 \\ O & O & O \\ O & O & \alpha I - \frac{1}{\alpha}B_1^T B_1 \end{pmatrix}. \tag{1.5}$$

The GRS preconditioner mentioned above is effective for solving large and sparse nonsymmetric saddle-point linear systems. However, there still exist some disadvantages in its implementation. From (1.5), we can see that both the (1, 3)-block and the (3, 3)-block in $\mathscr{R}_{GRS}$ tend to infinity as $\alpha \to 0_+$, while the (1, 3)-block tends to $-B_1$ and (3, 3)-block goes to infinity as $\alpha \to +\infty$. Therefore, we must choose appropriate parameter α to balance the diagonal and the off-diagonal parts, which is a difficult problem. In addition, the solutions of the systems of linear equations appeared in the Step 3 of Algorithm 1.1 are time consuming, since there exists $(\alpha I + C)^{-1}$ in the coefficient matrices of the linear systems.

In this paper we present a modified GRS (MGRS) preconditioner and a modified block-triangular splitting (MBTS) preconditioner by constructing a block-triangular splitting of coefficient matrix $\mathscr{A}$. The remainder of this paper is organized as follows. In Section 2, we present the MGRS preconditioner for generalized saddle-point linear systems (1.1) and analyze respectively the spectral property and the degree of the minimal polynomial of the MGRS preconditioned generalized saddle-point matrix. We develop the MBTS preconditioner and discuss the spectral distribution and the degree of the minimal polynomial of the MBTS preconditioned generalized saddle-point matrix in Section 3. We discuss the quasi-optimal parameters of the MGRS and MBTS preconditioners for two-dimensional Navier–Stokes equations in Section 4. In Section 5, we apply the MGRS and the MBTS preconditioners to three-dimensional Navier–Stokes equations and deduce the corresponding quasi-optimal parameters. Finally, in Section 6, numerical experiments are presented to show the effects of these new preconditioners.

2. The MGRS preconditioner and its spectral properties

According to the theory of preconditioning techniques [35], we know that the spectral distribution of the preconditioned matrix relates closely to the convergence of Krylov subspace methods and favorable convergence rates are often associated