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Effective implementation of the weak Galerkin finite element methods for the biharmonic equation

Lin Mu [a,*], Junping Wang [b], Xiu Ye [c]

[a] Computer Science and Mathematics Division Oak Ridge National Laboratory, Oak Ridge, TN, 37831, USA
[b] Division of Mathematical Sciences, National Science Foundation, Arlington, VA 22230, USA
[c] Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA

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ABSTRACT

The weak Galerkin (WG) methods have been introduced in Mu et al. (2013, 2014), [14] for solving the biharmonic equation. The purpose of this paper is to develop an algorithm to implement the WG methods effectively. This can be achieved by eliminating local unknowns to obtain a global system with significant reduction of size. In fact, this reduced global system is equivalent to the Schur complements of the WG methods. The unknowns of the Schur complement of the WG method are those defined on the element boundaries. The equivalence of the WG method and its Schur complement is established. The numerical results demonstrate the effectiveness of this new implementation technique.

1. Introduction

We consider the biharmonic equation of the form

$$\Delta^2 u = f, \quad \text{in } \Omega, \tag{1.1}$$

$$u = g, \quad \text{on } \partial\Omega, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = g_n \quad \text{on } \partial\Omega. \tag{1.3}$$

For the biharmonic problem (1.1) with Dirichlet and Neumann boundary conditions (1.2) and (1.3), the corresponding variational form is given by seeking $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = g$ and $\frac{\partial u}{\partial n}|_{\partial\Omega} = g_n$ such that

$$(\Delta u, \Delta v) = (f, v), \qquad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where $H_0^2(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial\Omega$.

Conforming finite element methods for this fourth order equation require finite element spaces to be subspaces of $H^2(\Omega)$ or $C^1(\Omega)$. Due to the complexity of construction of C^1 elements, H^2 conforming methods are rarely used in practice for solving the biharmonic equation. Due to this reason, many nonconforming or discontinuous finite element methods have been developed for solving the biharmonic equation [1–4]. Morley element [5,6] is a well known nonconforming element for the biharmonic equation for its simplicity. C^0 interior penalty methods were studied in [7,8]. In [9], a *hp*-version interior penalty discontinuous Galerkin (DG) methods were developed for the biharmonic equation.

* Corresponding author.
E-mail addresses: mul1@ornl.gov (L. Mu), jwang@nsf.gov (J. Wang), xxye@ualr.edu (X. Ye).

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Weak Galerkin methods refer to general finite element techniques for partial differential equations and were first introduced in [10] for second order elliptic equations. They are by designing using discontinuous approximating functions on general meshes to avoid construction of complicated elements such as C^1 conforming elements. In general, weak Galerkin finite element formulation can be derived directly from the variational form of the PDE by replacing the corresponding derivatives by the weak derivatives and adding a parameter independent stabilizer. Obviously, the WG method for the biharmonic equation should have the form

$$(\Delta_w u_h, \Delta_w v) + s(u_h, v) = (f, v), \tag{1.5}$$

where $s(\cdot, \cdot)$ is a parameter independent stabilizer. The WG formulation (1.5) in its primary form is symmetric and positive definite.

The main idea of weak Galerkin finite element methods is the use of weak functions and their corresponding weak derivatives in algorithm design. For the biharmonic equations, weak function has the form $v = \{v_0, v_b, v_n\}$ with $v = v_0$ inside of each element and $v = v_b$, $\nabla v \cdot \mathbf{n} = v_n$ on the boundary of the element. In the weak Galerkin method introduced in [11], v_0 and v_b are approximated by kth degree polynomials and v_n is approximated by the polynomial of degree $k - 1$. This method has been improved in [12] through polynomial degree reduction where v_b and v_n are both approximated by the polynomials of degree $k - 1$.

Introductions of weak functions and weak derivatives make the WG methods highly flexible. It also creates additional degrees of freedom associated with v_b and v_n. The purpose of this paper is to develop an algorithm to implement the WG methods introduced in [11–14] effectively. This can be achieved by deriving the Schur complements of the WG methods and eliminating the unknown u_0 from the globally coupled systems. Variables u_b and u_n defined on the element boundaries are the only unknowns of the Schur complements which significantly reduce globally coupled unknowns. We prove that the reduced system is symmetric and positive definite. The equivalence of the WG method and its Schur complement is also established. The results of this paper is based on the weak Galerkin method developed in [12]. The theory can also be applied to the WG method introduced in [11] directly.

The technique of eliminating the interior unknowns can also be found in hybridizable discontinuous Galerkin (HDG) method [15]. The HDG method is a close relative of the WG method. The WG method and HDG method are equivalent for the Poisson equation. However, the WG method differs from the HDG method for the second order elliptic equations with nonconstant diffusion coefficient and more sophisticated problems. These two methods are different fundamentally. The key element of HDG methods is numerical flux but the key element for WG method is weak derivatives. The concept of weak derivatives will make it easy for applying WG methods to solve different PDE problems by simply replacing derivatives in the weak formulations by weak derivatives.

The paper is organized as follows. A weak Laplacian operator is introduced in Section 2. In Section 3, we provide a description for the WG finite element scheme for the biharmonic equation introduced in [12]. In Section 4, a Schur complement formulation of the WG method is derived to reduce the cost in the implementation. Numerical experiments are conducted in Section 5.

2. Weak Laplacian and discrete weak Laplacian

Let T be any polygonal or polyhedral domain with boundary ∂T. A *weak function* on the region T refers to a function $v = \{v_0, v_b, v_n\}$ such that $v_0 \in L^2(T)$, $v_b \in H^{\frac{1}{2}}(\partial T)$, and $v_n \in H^{-\frac{1}{2}}(\partial T)$. The first component v_0 can be understood as the value of v in T and the second and the third components v_b and v_n represent v on ∂T and $\nabla v \cdot \mathbf{n}$ on ∂T, where $\mathbf{n}$ is the outward normal direction of T on its boundary. Note that v_b and v_n may not necessarily be related to the trace of v_0 and $\nabla v_0 \cdot \mathbf{n}$ on ∂K traces should be well-defined.

Denote by $\mathcal{W}(T)$ the space of all weak functions on T; i.e.,

$$\mathcal{W}(T) = \left\{ v = \{v_0, v_b, v_n\} : \ v_0 \in L^2(T), \ \ v_b \in H^{\frac{1}{2}}(\partial T), \ \ v_n \in H^{-\frac{1}{2}}(\partial T) \right\}. \tag{2.1}$$

Let $(\cdot, \cdot)_T$ stand for the L^2-inner product in $L^2(T)$, $\langle \cdot, \cdot \rangle_{\partial T}$ be the inner product in $L^2(\partial T)$. For convenience, define $G^2(T)$ as follows:

$$G^2(T) = \{\varphi : \ \varphi \in H^1(T), \Delta\varphi \in L^2(T)\}.$$

It is clear that, for any $\varphi \in G^2(T)$, we have $\nabla\varphi \in H(div, T)$. It follows that $\nabla\varphi \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\partial T)$ for any $\varphi \in G^2(T)$.

Definition 2.1 (*Weak Laplacian*)**.** The dual of $L^2(T)$ can be identified with itself by using the standard L^2 inner product as the action of linear functionals. With a similar interpretation, for any $v \in \mathcal{W}(T)$, the *weak Laplacian* of $v = \{v_0, v_b, v_n\}$ is defined as a linear functional $\Delta_w v$ in the dual space of $G^2(T)$ whose action on each $\varphi \in G^2(T)$ is given by

$$(\Delta_w v, \ \varphi)_T = (v_0, \ \Delta\varphi)_T - \langle v_b, \ \nabla\varphi \cdot \mathbf{n}\rangle_{\partial T} + \langle v_n, \ \varphi\rangle_{\partial T}, \tag{2.2}$$

where $\mathbf{n}$ is the outward normal direction to ∂T.

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