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Filter regularization for final value fractional diffusion problem with deterministic and random noise

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ABSTRACT

In this paper, we consider an inverse problem for a time fractional diffusion equation with inhomogeneous source to determine the initial data from the observation data provided at a later time. In general, this problem is ill-posed, therefore we construct a regularized solution using the filter regularization method in both cases: the deterministic case and random noise case. First, we propose both parameter choice rule methods, the *a-priori* and the *a-posteriori* methods. Then, we obtain the convergence rates and provide examples of filters. We also provide a numerical example to illustrate our results.

1. Introduction

In this work, we consider the problem of recovering the distribution $u(x, 0)$ from a final measure of $u(x, T)$ for the following inhomogeneous time-fractional diffusion problem

$$\begin{cases} \partial_t^\alpha u = Au(x,t) + F(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $(d = 1, 2, 3)$ with sufficiently smooth boundary $\partial\Omega$, and $T > 0$ is a given number. The source function $F \in L^\infty(0, T; L^2(\Omega))$ and the final data $g \in L^2(\Omega)$ are given. In the model (1.1), $\partial_t^\alpha u$ refers to the Caputo derivative of order α, $(0 < \alpha < 1)$, and it is defined by

$$\partial_t^\alpha u := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds,$$

where $\Gamma(x)$ denotes the standard Gamma function. The operator A is a symmetric uniformly elliptic operator. Note that if the fractional order α tends to unity, the fractional derivative $\partial_t^\alpha u$ becomes the first-order derivative $\partial_t u$, and thus problem

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(1.1) reproduces the standard parabolic equation. See, e.g., [1], for the definition and properties of Caputo's derivative. The fractional diffusion equation (1.1) has found a number of important practical applications, such as the dynamics of viscoelastic materials, and contaminant transport within underground water flow [1–4].

The direct problems, i.e., the initial value problem and the initial boundary value problem for fractional diffusion equation, have been studied extensively in the past few years [5–7]. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. In situation we only may know the noisy data on a part of the boundary or at some interior points of the considered domain, which leads to an inverse problem. Many authors have been studied inverse problems for fractional diffusion equations, for example, R. Gorenflo et al. [8], M. Meerschaert [9], W. Rundell et al. [10,11], W. Clean [6], M. Kirane et al. [12–14], M. Yamamoto et al. [15,16], K. Mustafa [7].

As it is known, problem (1.1) is ill-posed in the sense of Hadamard [17], i.e., its solution may not exist, and if it exists, it does not depend continuously on the given data. In fact, small noise on the measured data may lead to solutions with large errors. This makes the numerical computation difficult, hence a regularization process is needed.

When $\alpha = 1$, the problem (1.1) is a classical problem and has been studied actively in the past 30 years using many regularization methods, see [18] and the references therein. However, there are not many studies on the fractional backward problem (1.1) for $\alpha \in (0, 1)$. For the homogeneous case of backward problem, i.e., $F = 0$ in (1.1), there are many works. For example, Sakamoto and Yamamoto [19] have proved that there exists a unique weak solution for this backward problem; Liu et al. [20] used a quasi-reversibility method to solve the homogeneous backward problem in one-dimensional case for special coefficients; and very recently, it has been considered by some other authors, such as [21–23]. Since the fractional derivative is nonlocal, the backward problems for such models become much more difficult due to the following two reasons. First, the backward problems for diffusion processes are ill posed, corresponding to the irreversibility of time; secondly, the strong smoothing effect of the forward diffusion process makes it very difficult to reconstruct the possible discontinuities arising in the initial status. The backward problem for homogeneous time fractional diffusion equation has been also considered in some recent works, for example [24–26]. However, the results for inhomogeneous case are very rare. Very recently, Mohammad F. Al-Jamal [27,28]considered the backward problem for inhomogeneous case for the source $F = F(x)$. The deterministic problem in the case of $F := F(x, t)$ has been recently studied in [29,30].

In the present paper, we are interested in considering the backward problem in both cases: the deterministic case and random noise case. The deterministic case has been studied in many works. However, the results for random noise case for backward fractional diffusion are very limited. Motivated by these reasons, our aim here in this paper is to estimate (or reconstruct) the unknown $u(x, 0)$ by use of the two following observations:

For the deterministic case. The functions (g, F) are approximated by the noisy observation data $(g^\varepsilon, F^\varepsilon)$, such that

$$\|g^\epsilon - g\|_{L^2(\Omega)} \le \epsilon, \qquad \|F^\epsilon - F\|_{L^\infty(0,T;L^2(\Omega))} \le \epsilon,$$

where $\epsilon > 0$ is a small positive number, used for measuring the noise level.

For random noise case. The function (g^ϵ, F^ϵ) is replaced by $(\widetilde{g}_\epsilon, \widetilde{F}_\epsilon)$ where $\widetilde{g}_\epsilon$ and $\widetilde{F}_\epsilon$ have the following observation model

$$\widetilde{g}_\epsilon(x) = g(x) + \epsilon\xi(x), \qquad \widetilde{F}_\epsilon(x, t) = F(x, t) + \epsilon\psi(x),$$

where ξ and ψ are stochastic errors which are defined in Section 4, ϵ corresponds to the noise level. Such above random noise has been studied in some other papers, for example [31,32].

In the present paper, we mainly propose a priori and a posteriori regularization parameter choice rules using a new general filter regularization (GFR). The filter regularization has been developed by many authors, such as D. Lesnic et al. [33], N.H. Tuan et al. [34], M. Kirane et al. [35], T. Wei et al. [36]. Our method may be better than other already existing regularization methods because some previous methods such as quasi-boundary value method and truncation methods can be deduced from GFR by choosing some suitable filters. Moreover, we can improve the convergence rate by comparing some different filters. The other motivation comes from the random model where we can use filter method. The truncation method in [37], the generalized Tikhonov equation in [24], quasi-boundary value method in [23], etc. are some special case of our method. We present the convergence rate under the a priori bound assumption on the exact solution and a priori parameter choice rule. In practice, it is not easy to obtain the a priori bound. Therefore, we need to obtain a convergence rate under the a posteriori parameter choice rule which is independent of the a priori bound. This can be seen as an extension of the works in [20–23,25,38].

The manuscript is organized as follows. A new general filter regularization method is introduced in Section 2. In Section 3, the error estimate is obtained with the a priori parameter choice rule. The a posteriori parameter choice rule is given in Section 4, which also leads to a Hölder-type error estimate. In Section 5, the theoretical foundation of the general filter regularization (GFR) method and seven regularization filters deduced from the GFR are presented. We also give one numerical example which shows the effectiveness of our method.

2. A new general filter regularization method

For the sequel, we introduce the two-parameters Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$