ARTICLE IN PRESS

Computers and Mathematics with Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A robust higher order compact scheme for solving general second order partial differential equation with derivative source terms on nonuniform curvilinear meshes

Swapan K. Pandit *, Anirban Chattopadhyay

Integrated Science Education and Research Centre (ISERC), Visva-Bharati, Santiniketan, West Bengal-731 235, India

ARTICLE INFO

Article history:
Received 13 November 2016
Received in revised form 10 May 2017
Accepted 17 June 2017
Available online xxxx

Keywords:
Compact scheme
Convection–diffusion equation
N–S equations
Nonorthogonal curvilinear grid
Distorted grid

ABSTRACT

A fourth order compact finite difference scheme is proposed for solving general second order steady partial differential equation (PDE) in two-dimension (2D) on geometries having nonuniform curvilinear grids. In this work, the main efforts are focused not only on nonorthogonal curvilinear grids but also on the presence of mixed derivative term and nonhomogeneous derivative source terms in the governing equation. This is in turn suitable for solving fluid flow and heat transfer problems governed by Navier–Stokes (N–S) equations on geometries having nonuniform and nonorthogonal curvilinear grids. The newly proposed scheme has been applied to solve general second order partial differential equation having analytical solution and some pertinent fluid flow problems, namely, viscous flows in a lid driven cavity such as trapezoidal cavity using nonorthogonal grid, square cavity using distorted grid, complicated enclosures using curvilinear grid, and mixed convection flow in a bottom wavy wall cavity. It is seen to efficiently capture steady-state solutions of the N–S equations with Dirichlet as well as Neumann boundary conditions. Detailed comparison data produced by the proposed scheme for all the test cases are provided and compared with existing analytical as well as established numerical results available in the literature. Excellent comparison is obtained in all the cases.

1. Introduction

The second order partial differential equations with nonhomogeneous derivative source terms are of common occurrence in mathematical physics and several other areas of science and engineering such as option pricing in stochastic volatility models, flow and heat transfer through porous media and in numerical mathematics. Moreover, convection–diffusion-reaction equations under coordinate transformation on geometries having nonorthogonal grids produce a general second order PDE as the mathematically generated mixed derivative term appeared to the transformed equations. Numerical prediction of these kinds of equations plays a very important role in computational fluid dynamics (CFD) to simulate fluid flow problems. In a nutshell all these PDEs can be categorized under the same umbrella of general second order PDE with or without nonhomogeneous source terms. Therefore, accurate, stable and efficient difference representations of the convection–diffusion-reaction equations are of vital importance. In particular, the second order central difference and the upwind schemes have been the most popular ones because of their straightforwardness in application. For problems having well-behaved solutions, they often yield quite good results on reasonable meshes while the solution may be of poor quality for convection dominated flows if the mesh is not sufficiently refined. On the other hand, higher order discretization is

* Corresponding author.
E-mail addresses: swapankumar.pandit@visva-bharati.ac.in (S.K. Pandit), animath81@rediffmail.com (A. Chattopadhyay).

http://dx.doi.org/10.1016/j.camwa.2017.06.031

Please cite this article in press as: S.K. Pandit, A. Chattopadhyay, A robust higher order compact scheme for solving general second order partial differential equation with derivative source terms on nonuniform curvilinear meshes, Computers and Mathematics with Applications (2017), http://dx.doi.org/10.1016/j.camwa.2017.06.031.

generally associated with large non-compact stencils which increase the band-width of the resultant coefficient matrix. As a consequence, the sparsity of this matrix decreases resulting in a reduced rate of convergence of the iterative solver. However, both the mesh refinement and increased matrix band-width i.e. decreased sparsity of the matrix invariably lead to a large number of arithmetic operations. Thus, neither a lower order accurate scheme on a fine mesh nor a higher order accurate one on a noncompact stencil seems to be computationally cost effective. In this regard, higher order compact (HOC) finite difference schemes become important. Compact schemes characterize high accuracy solution on smaller stencils, so cost effective with favorable numerical stability. In addition to these, Dirichlet and Neumann boundary conditions can be used with less effort. In the spirit of accuracy and compactness, the high order compact schemes have seen increasing popularity in solving computational and geophysical fluid dynamics problems over the past few decades. There are different approaches to develop HOC schemes. One of the approaches to achieve higher-order compactness [1–3] is based on Padé approximation [4], which is an implicit relation between the derivatives and the functions at a nodal point around which the differences are taken. It has become very popular and has been extensively used to solve problems of fluid dynamics [3,5–7] and wave propagation [8,9]. These schemes exhibit spectral like resolution.

By combining virtues of compact discretization and Padé approximation, Sen [10] has developed a new family of implicit HOC schemes for unsteady convection–diffusion equation with variable convection coefficients. With this philosophy in mind Sen [11] proposed a generalized HOC formulation for parabolic problem with mixed derivative. In the recent past, with the efforts to increase computational efficiency of HOC schemes, the development and implementation of HOC-ADI algorithms [12–15] combine computational efficiency of ADI approach and high-order accuracy of HOC schemes.

In recent years, one of the other approaches is to use original differential equation to achieve higher order compactness and has generated renewed interest to study incompressible viscous flows. Considering this idea a variety of specialized techniques have been developed so far. In the work of [16], the authors derived a fourth order compact finite difference scheme for 2D elliptic PDE with mixed derivative by considering the PDE itself as an auxiliary relation. Their work is limited by considering diffusion coefficients as 1 and the coefficient of mixed derivative is less than 4. In an another work, Karaa [17] solved 2D elliptic and parabolic equations with mixed derivative having variable coefficient by developing a fourth order compact finite difference scheme using polynomial approximation, but was again restricted the values of the diffusion coefficients as 1. However, out of the plethora of HOC schemes [12–24] designed so far for discretizing convection–diffusion equation, a few can tackle the generalized second order PDE. Majority of these HOC approaches on 9-point compact stencil are confined to uniform space grids. As such these schemes could not fully exploit the advantages associated with using non-uniform grids, particularly that of mesh grading to resolve smaller scales in the regions of large gradients in the physical domain. However, analysis of the developed schemes from the perspective of nonorthogonality properties of grids was not studied, instead attention was primarily focused upon orthogonal grids on rectangular geometries. Recently, Kalita et al. [25], Mancera [26], Spotz and Carey [27], Wang et al. [28] and Ge and Cao [29] have developed some HOC schemes on nonuniform grids for the 2D convection–diffusion equations. Perusal of the literature of these kinds show that there exist many works on the higher-order finite difference methods but a few have addressed on the generalized convection–diffusion–reaction equation with or without mixed derivatives on curvilinear grids [30]. Off late, Pandit et al. [31,32] proposed higher order compact scheme on geometries beyond rectangular. It is worthy to mention here that the developed higher order compact scheme is strictly limited to its use with orthogonal curvilinear grids. However, for a wide range of problems of practical relevance, the grid can be non orthogonal also.

The objective of the present work is twofold: (i) generalized formulation of the class of HOC schemes for nonorthogonal curvilinear grids, and (ii) tackling of mixed derivative terms and nonhomogeneous derivative terms in the governing equations via Padé approximations. To the best of authors' knowledge, no studies have been found in the literature on the formulation of HOC schemes using 9-point stencils with nonorthogonal curvilinear grids. The focus of this investigation is to develop a general scheme that can be used for a wide range of problems. This work on the HOC schemes is motivated by the need to solve accurately boundary value problems on nonuniform nonorthogonal curvilinear grids. However, for general curvilinear geometries, it is often assumed that the grid spacing is uniform in the computational domain obtained by the mapping of nonuniform curvilinear grids in the physical domain.

This paper is organized as follows: Section 2 presents proposed methodology in detail; Section 3 discusses algorithmic implementations; Section 4 demonstrates accuracy and robustness of the proposed algorithms by means of a variety of numerical results. Concluding remarks, finally, are presented in Section 5.

2. Basic formulations and discretization procedure

The general second order steady partial differential equation for a transport variable $\phi(x, y)$ defined in some continuous domain $\Omega \subset \mathbb{R}^2$ with nonhomogeneous derivative source terms and suitable boundary conditions can be written in nondimensional form as:

$$\begin{cases} \alpha \dfrac{\partial^2 \phi}{\partial x^2} + \tau \dfrac{\partial^2 \phi}{\partial x \partial y} + \beta \dfrac{\partial^2 \phi}{\partial y^2} + \gamma \dfrac{\partial \phi}{\partial x} + \nu \dfrac{\partial \phi}{\partial y} + \omega \phi = \chi + l_1 \dfrac{\partial \theta}{\partial x} + l_2 \dfrac{\partial \theta}{\partial y}, & (x, y) \in \Omega \\ b_1(x, y)\phi + b_2(x, y)\dfrac{\partial \phi}{\partial n} = b_3(x, y), & (x, y) \in \partial\Omega \end{cases} \tag{1}$$