Computers and Mathematics with Applications ■ (■■■■) ■■■–■■■

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Two-grid mixed finite element method for nonlinear hyperbolic equations

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ARTICLE INFO

Article history:
Received 11 May 2016
Received in revised form 5 June 2017
Accepted 17 June 2017
Available online xxxx

Keywords:
Nonlinear hyperbolic equations
Two-grid method
Mixed finite element method
Error estimate

ABSTRACT

In this paper, a combination method of mixed finite element method and two-grid scheme is constructed for solving numerically the two-dimensional nonlinear hyperbolic equations. In this approach, the nonlinear system is solved on a coarse mesh with width H, and the linear system is solved on a fine mesh with width $h \ll H$. It is proved that the coarse grid can be much coarser than the fine grid. The two-grid methods achieve asymptotically optimal approximation as long as the mesh sizes $H = \mathcal{O}(h^{\frac{1}{2}})$ in the first algorithm and $H = \mathcal{O}(h^{\frac{1}{4}})$ in the second algorithm, respectively. Error estimates are derived in detail. Finally, two numerical examples are carried out to verify the accuracy and efficiency of the presented method.

1. Introduction

Hyperbolic equations have appeared in quite a few physical fields, such as mathematical models the vibrations of structures, the sound wave, the electromagnetic wave and so on. In recent years, hyperbolic equations have received a great deal of attention and research. Dupont [1] has analyzed both the semidiscrete and a fully discrete Galerkin method for second hyperbolic equations, and improved $L^\infty(L^2)$-error estimates for second order linear hyperbolic equations were derived by Baker [2]. The estimates were based on the energy inequality and a nonstandard type of energy relation, respectively. Geveci [3] considered mixed finite element methods for the wave equation. Cowsar et al. have demonstrated the stability of the discrete-time mixed finite element schemes for the wave equation in [4]. Jenkins et al. have obtained a priori $L^\infty(L^2)$-error estimates for mixed finite element displacement formulations of the acoustic wave equation in [5]. Chen et al. [6,7] have derived a priori error estimates for the semidiscrete and the fully discrete mixed finite method for nonlinear hyperbolic equations. Thus, the development of an effective computational method for investigating hyperbolic problems has practical significance.

The two-grid method is a highly efficient and accurate method and has been well developed in the past decades, which was first introduced by Xu [8,9] as a discretization techniques for nonsymmetric and nonlinear elliptic problems. With the development of the technique, the method was applied and investigated to other problems by many authors. For example, Zhou et al. [10] for Maxwell eigenvalue problems, Weng et al. [11] for Steklov eigenvalue problem, Liu et al. [12] for nonlinear non-Fickian flow model in porous media, Holst et al. [13] for semilinear interface problems, Ignat and Zuazua [14] for nonlinear Schrödinger equations, Dawson and Wheeler [15] for nonlinear parabolic equations, Chen et al. [16] present a detailed analysis of two-grid finite element method approximation to second-order nonlinear hyperbolic equations. Moreover, Wu and Allen [17] studied the two-grid discretization scheme of conforming expanded mixed finite element

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http://dx.doi.org/10.1016/j.camwa.2017.06.022

method for reaction–diffusion equations. Especially, Chen et al. [18,19] have constructed and analyzed a two-steps algorithm by using two-grid idea for mixed finite element solution of nonlinear reaction–diffusion equations. So we extend the two-grid discretization scheme to solving the hyperbolic problems.

Influenced by the work mentioned above, the purpose of this paper is to formulate two two-grid algorithms for mixed finite element approximations of second-order nonlinear hyperbolic equations and analysis the convergence property between the exact solution and the numerical solution. Comparing with [6], we can verify two-grid algorithms are effective for saving a large amount of computational time and without losing accuracy as long as the mesh sizes satisfy $H = \mathcal{O}(h^{\frac{1}{2}})$ in the first algorithm and $H = \mathcal{O}(h^{\frac{1}{4}})$ in the second algorithm, respectively. We are interested in the following second order nonlinear hyperbolic equation:

$$c(u)u_{tt} - \nabla \cdot (\boldsymbol{K}\nabla u) = f, \quad (\boldsymbol{x}, t) \in \Omega \times J, \tag{1.1}$$

with initial conditions

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,\ t = 0, \tag{1.2}$$

and boundary condition

$$\boldsymbol{K}\nabla u \cdot \boldsymbol{\nu} = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times J, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a convex polygonal domain, $\boldsymbol{\nu}$ is the unit exterior normal to $\partial\Omega$, $J = (0, T]$, $\boldsymbol{K} : \Omega \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ is a symmetric and uniformly positive definite matrix, u_{tt} and u_t denote $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial u}{\partial t}$, respectively. To state some results about the convergence of the present method, we will assume the following conditions:

(i) We assume that the function $c(u)$ is bounded above and below by positive constants

$$c_* \le c(u) \le c^*.$$

(ii) We also assume that function $\boldsymbol{\kappa} = \boldsymbol{K}^{-1}$ is a square-integrable, symmetric, uniformly positive definite tensor defined on Ω. Additionally, we assume that there exist constants $K_*, K^* > 0$, for every $\boldsymbol{x} \in \bar{\Omega}$ and any vector $\boldsymbol{y} \in \mathbb{R}^2$, such that

$$K_* |\boldsymbol{y}|^2 \le \boldsymbol{y}^T \boldsymbol{\kappa}(\boldsymbol{x}) \boldsymbol{y} \le K^* |\boldsymbol{y}|^2.$$

(iii) The functions $c = c(u), f$ are continuously differentiable with respect to t and u . Moreover, there exists a bound K_1, such that

$$|f|, \qquad \left|\frac{\partial f}{\partial t}\right|, \qquad \left|\frac{\partial c}{\partial u}\right|, \qquad \left|\frac{\partial^2 c}{\partial u^2}\right| \le K_1.$$

(iv) For some integer $r \ge 1$, we assume that the solution u of (1.1) satisfies

$$u \in L^2(H^{r+2}(\Omega) \cap W^{r,\infty}(\Omega)), \qquad \frac{\partial u}{\partial t} \in L^2(H^{r+2}(\Omega) \cap W^{r,\infty}(\Omega)),$$
$$\frac{\partial^2 u}{\partial t^2} \in L^2(H^{r+1}(\Omega)), \qquad \frac{\partial^4 u}{\partial t^4} \in L^\infty(L^2(\Omega)).$$

The rest of the paper is organized as follows: In Section 2, some notations, projection operators and some known properties of these projections are introduced. In Section 3, the fully discrete mixed finite element scheme and some lemmas useful for establishing the convergence results are stated. In Section 4, error estimates and convergence results are presented for solutions of the two-grid method. Some simple numerical examples are used to confirm the theoretical results in Section 5. Finally, some conclusions and further works are presented in the last section.

Throughout this paper, let C denote a generic positive constant that is independent of the spatial or time discretization parameters and may be different at its different occurrences.

2. Some notation and projections

Let $L^p(\Omega)$ for $p \ge 1$ denote the standard Banach space defined on Ω, with norm $\|\cdot\|_p$. We shall also use the standard Sobolev space $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{m,p}$ given by $\|\phi\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha \phi\|_{L^p(\Omega)}^p$. To simplify the notation, for $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product, scalar and vector.

Let

$$H(div; \Omega) = \{\boldsymbol{v} : \boldsymbol{v} \in (L^2(\Omega))^2,\ \nabla \cdot \boldsymbol{v} \in L^2(\Omega)\},$$
$$\mathbf{V} = \{\boldsymbol{v} \in H(div; \Omega) : \boldsymbol{v} \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0\},$$
$$W = L^2(\Omega),$$

equipped with the norm given by:

$$\|\boldsymbol{v}\|_{H(div;\Omega)} \equiv (\|\boldsymbol{v}\|^2 + \|\nabla \cdot \boldsymbol{v}\|^2)^{1/2}.$$