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On the approximation of electromagnetic fields by edge finite elements. Part 2: A heterogeneous multiscale method for Maxwell's equations

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ABSTRACT

In the second part of this series of papers we consider highly oscillatory media. In this situation, the need for a triangulation that resolves all microscopic details of the medium makes standard edge finite elements impractical because of the resulting tremendous computational load. On the other hand, undersampling by using a coarse mesh might lead to inaccurate results. To overcome these difficulties and to improve the ratio between accuracy and computational costs, homogenization techniques can be used. In this paper we recall analytical homogenization results and propose a novel numerical homogenization scheme for Maxwell's equations in frequency domain. This scheme follows the design principles of heterogeneous multiscale methods. We prove convergence to the effective solution of the multiscale Maxwell's equations in a periodic setting and give numerical experiments in accordance to the stated results.

1. Introduction

As in the first part [1] of the series entitled "On the Approximation of Electromagnetic Fields by Edge Finite Elements" we study the numerical approximation of electromagnetic fields governed by Maxwell's equations. Here, we are interested in materials that oscillate on a microscopic length scale η much smaller than the size of the computational domain. Such materials are modeled by their electric permittivity tensor ε^η and their magnetic permeability tensor μ^η. The microscopic nature of these materials properties are indicated by the superscript η.

Our goal is to approximate the macroscopic behavior of the electric field $\boldsymbol{e}^\eta$. To obtain a reliable approximation of it using standard edge finite elements, the microscopic details in μ^η and ε^η need to be resolved. This leads to an enormous number of degrees of freedom and might result in infeasibly high computational costs. Therefore, more involved methods are needed. A standard approach consists in replacing the multiscale tensors μ^η and ε^η with effective ones not depending on the microscale, such that the macroscopic properties of the unknown electric field remain unchanged. While mixing formulas, see [2] and the references therein, are often used in physics and engineering, homogenization results are more common in the mathematical literature. In the seminal book [3] homogenization results for elliptic equations involving a **curl**-**curl** operator were proven. This proof relies on a div-**curl** lemma and uses the technique of compensated compactness. Homogenization results of time-dependent Maxwell's equations can be found e.g. in [4] and in [5,6]. In the later references

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the notion of two-scale convergence was used to justify rigorously the homogenization process. Using similar techniques in a frequency domain setting, homogenization for an electromagnetic scattering problem in the whole space by a multiscale obstacle was achieved in [7]. The interest from a computational point of view of homogenizing microstructure inhomogeneities to avoid the need for excessive grid refinement has often been pointed out, see this non exhaustive list of references for instance [8–13]. In this article, we give a slightly different homogenization result based on two-scale convergence in the second part of Section 2. This is the basis upon which our numerical homogenization scheme is built.

The term "numerical homogenization" is used for numerical methods that approximate the effective or homogenized solution of a multiscale equation without knowing the exact effective parameters. The most fundamental numerical homogenization methods precompute numerically effective parameters not depending on the microscale. In a second step, an approximated effective equation can be solved with standard methods. In [14] two such methods for Maxwell's equations are compared. The first one is based on the homogenization results as described above, while the second one is based on a Floquet–Bloch expansion, see [15]. While such methods are quite accessible, their range of application is usually limited to periodic settings without straightforward generalizations to more involved settings. In addition the influence of the numerical discretization error in the approximation of the effective parameters on the overall solution is hardly ever considered.

Novel numerical homogenization schemes, such as e.g. Multiscale Finite Element Methods (MsFEM) [16,17] and Heterogeneous Multiscale Method (HMM) [18,19] do not rely on a priori computation of the effective parameters to approximate the effective behavior. In this paper we follow the HMM framework to propose a novel numerical homogenization scheme for Maxwell's equations. A detailed description of our scheme can be found in Section 3. In [20,21] an HMM scheme was applied to an Eddy current problem, but without rigorous convergence proof. There, the macro problem, which involves the **curl**-operator, was discretized with Lagrange finite elements and the introduction of a stabilization term was needed. By contrast, we use edge finite elements for the macro solver and prove an abstract a priori error bound in Section 4. Very recently another HMM scheme for Maxwell's equations was proposed in [22], where in similarity with our scheme, edge finite elements are used for the macro solver. Nevertheless, the two methods differ from each other. First of all, in [22] a problem with non-vanishing conductivity was considered, whereas in this article the conductivity equals zero. As a consequence, the Maxwell's equations are not coercive in our case. To overcome this additional difficulty we use the notion of T-coercivity [23,24]. Secondly, their method relies on a divergence regularization and thus the micro problem for the permeability differs from the micro problems we use. We will highlight the similarities and differences of these two HMM methods in more detail in the course of this article. In Section 5 we illustrate how to apply the general method and the error bounds to concrete settings. Numerical experiments corroborating the theoretical results are presented in Section 6.

1.1. Notation

Whenever possible we follow the notations of [1]. In the following we repeat the most important ones for the self-containment of this article and add some complementary ones. Let $\mathcal{O} \subset \mathbb{R}^3$ be an open, bounded connected set with Lipschitz boundary $\partial\mathcal{O}$ in $\mathbb{R}^3$, i.e. $\mathcal{O}$ is a domain. The standard orthonormal basis of $\mathbb{R}^3$ is denoted by $(\boldsymbol{e}_k)_{k=1,2,3}$ and the $d \times d$-identity matrix (with $d = 2$ or 3) by I_d.

For a sufficiently smooth vector valued function $\boldsymbol{v}$, we write $\mathbf{curl}\,\boldsymbol{v}$ to denote its curl. If the function depends on two variables and the curl is only taken with respect to one of them, we indicate this using subscripts. E.g. for $\boldsymbol{v} : (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})$ we write $\mathbf{curl}_{\boldsymbol{x}}\boldsymbol{v}$, resp. $\mathbf{curl}_{\boldsymbol{y}}\boldsymbol{v}$, for the curl of $\boldsymbol{v}$ taken with respect to the first, resp. the second variable. For other differential operators we use the same notation, e.g. $\mathrm{div}_{\boldsymbol{y}}\,\boldsymbol{v}$ denotes the divergence of $\boldsymbol{v}$ with respect to its second argument.

We use the usual notation $H^\ell(\mathcal{O})$ for Sobolev spaces with the standard convention that $L^2(\mathcal{O}) = H^0(\mathcal{O})$. In addition we will denote their vector-valued counterparts in bold face, e.g. $\boldsymbol{H}^\ell(\mathcal{O}) := \left(H^\ell(\mathcal{O})\right)^3$. By $(\cdot|\cdot)_{\ell,\mathcal{O}}$, respectively $\|\cdot\|_{\ell,\mathcal{O}}$, we denote the standard scalar product, resp. the standard norm in $H^\ell(\mathcal{O})$ or $\boldsymbol{H}^\ell(\mathcal{O})$. Furthermore, we will use the notation $\boldsymbol{H}(\mathbf{curl}; \mathcal{O})$ for $\boldsymbol{L}^2(\mathcal{O})$-measurable functions whose curl lays in the same function space. Its norm is given by

$$\|\boldsymbol{v}\|_{\mathbf{curl},\mathcal{O}} = \Big((\boldsymbol{v}|\boldsymbol{v})_{0,\mathcal{O}} + (\mathbf{curl}\,\boldsymbol{v}|\mathbf{curl}\,\boldsymbol{v})_{0,\mathcal{O}}\Big)^{1/2}.$$

The subspace of functions in $\boldsymbol{H}(\mathbf{curl}; \mathcal{O})$ with vanishing tangential component on the boundary $\partial\mathcal{O}$ is denoted by $\boldsymbol{H}_0(\mathbf{curl}; \mathcal{O})$. Note that $\boldsymbol{H}_0(\mathbf{curl}; \mathcal{O})$ is the closure of $\boldsymbol{\mathcal{D}}(\mathcal{O})$ in $\boldsymbol{H}(\mathbf{curl}; \mathcal{O})$. While for a cuboidal domain $\mathcal{O}$ periodic boundary conditions are widely used for Sobolev spaces, they are less common for $\boldsymbol{H}(\mathbf{curl}; \mathcal{O})$. Similarly to $\boldsymbol{H}_0(\mathbf{curl}; \mathcal{O})$, we let $\boldsymbol{H}_{\mathrm{per}}(\mathbf{curl}; \mathcal{O})$ be the closure of $\boldsymbol{C}^\infty_{\mathrm{per}}(\mathcal{O})$ in $\boldsymbol{H}(\mathbf{curl}; \mathcal{O})$, where $\boldsymbol{C}^\infty_{\mathrm{per}}(\mathcal{O})$ denotes the space of smooth $\mathcal{O}$-periodic functions restricted to $\mathcal{O}$. We proceed similarly to define $\boldsymbol{H}(\mathrm{div}; \mathcal{O})$ for $\boldsymbol{L}^2(\mathcal{O})$-measurable functions whose divergence lays in the same function space, and the ad hoc subspaces $\boldsymbol{H}_0(\mathrm{div}; \mathcal{O})$ and $\boldsymbol{H}_{\mathrm{per}}(\mathrm{div}; \mathcal{O})$ (replace tangential component by normal component in the previous definition). In this article, periodic boundary conditions are mainly used for the centered unit cube $Y = (-1/2, 1/2)^3$ and its shifted and scaled version given by

$$Y_\delta(\boldsymbol{x}) := \boldsymbol{x} + (-\delta/2, \delta/2)^3$$

for $\delta > 0$ and $\boldsymbol{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$.