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On determinants of cyclic pentadiagonal matrices with Toeplitz structure

Jiteng Jia[a,*], Sumei Li[b]

[a] School of Mathematics and Statistics, Xidian University, Xi'an, Shaanxi 710071, China
[b] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China

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ABSTRACT

Cyclic pentadiagonal matrices with Toeplitz structure have received tremendous attention in recent years. In the current paper, we present a block upper triangular transformation of the cyclic pentadiagonal Toeplitz matrices. By using the transformation, the determinant of an n-by-n cyclic pentadiagonal Toeplitz matrix can be readily evaluated since one just needs to compute the determinant of a 4-by-4 matrix obtained from the transformation. In addition, an efficient numerical algorithm of $O(n)$ is derived for computing nth order cyclic pentadiagonal Toeplitz determinants. Some numerical experiments are given to show the performance of the proposed algorithm.

1. Introduction and objectives

In the current paper, we mainly consider the determinant of a cyclic pentadiagonal Toeplitz matrix of the form

$$A = \begin{bmatrix} d & a & c & & & e & b \\ b & d & a & c & & & e \\ e & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & c \\ c & & & e & b & d & a \\ a & c & & & e & b & d \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{1.1}$$

where $a, b, c, d, e \in \mathbb{R}$, and we assume that $c \neq 0$ and $n \geq 5$. In fact, similar results can be obtained by the use of the argument given in Section 2 if $c = 0$.

From practical point of view, pentadiagonal matrices and cyclic pentadiagonal matrices frequently arise from boundary value problems (BVPs) involving 4th-order derivatives [1,2], and, computational algorithms for the determinants are linked to the problem of obtaining efficient test for the existence of unique solutions of the PDEs [3,4]. Usually, the determinant of a square matrix can be computed by the well-known Laplace formula [5]. However, algorithms based on such formula for

* Corresponding author.
E-mail address: lavenderjjt@163.com (J. Jia).

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large matrices may be time-consuming due to the number of required operations grows quickly. Therefore, more involved techniques have been developed for the determinant evaluation.

In order to evaluate the determinant of the cyclic pentadiagonal matrices, some authors have devised numerical (or symbolic) algorithms recently, see [6–8]. The main objective of this paper is to develop an efficient numerical algorithm for the determinant of an n-by-n cyclic pentadiagonal Toeplitz matrix. Recent developments of algorithms for the determinants and permanents of other related matrices, see e.g. [9–16].

The present paper is organized as follows. The main results are given in the next section. In Section 3, we give the results of some numerical experiments to show the performance of our algorithm. Finally we make some concluding remarks in Section 4.

2. Main results

In this section, we first present an approach for the determinant of a cyclic pentadiagonal Toeplitz matrix. Based on some elementary optimization methods, we then improve the proposed approach and develop an efficient numerical algorithm with the cost of $7n + O(\log n)$ for evaluating nth order cyclic pentadiagonal Toeplitz determinants.

2.1. An approach based on block upper triangular transformation

Let $\{C_1, C_2, \dots, C_n\}$ be the column vectors of the matrix A as in (1.1). If we successively interchange the first two columns C_1 and C_2 with other columns to make them as the last two columns, we then obtain a nearly lower triangular Toeplitz matrix T with the column vectors $\{C_3, C_4, \dots, C_n, C_1, C_2\}$, thus

$$T = \begin{bmatrix} c & & & e & b & d & a \\ a & c & & & e & b & d \\ d & \ddots & \ddots & & & e & b \\ b & \ddots & \ddots & \ddots & & & e \\ e & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & c & \\ & & e & b & d & a & c \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{2.1}$$

Since we made $2n - 2$ column interchanges in the matrix reordering above, we readily obtain the following relation:

$$\det(A) = \det(T).$$

We now consider transforming the matrix T (2.1) into a 2-by-2 block upper triangular matrix of the form:

$$U = \begin{bmatrix} c \cdot I_{n-4} & U_{12} \\ \mathbf{0} & U_{22} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{2.2}$$

where U_{12} and U_{22} are matrices of size $(n-4)$-by-4 and 4-by-4, respectively. Below, we show that the above matrix can be obtained from the original matrix T and a suitable choice $\omega_{i+1,i}, \omega_{i+2,i}, \omega_{i+3,i}$, and $\omega_{i+4,i}$ of the following matrices:

$$\Omega_1 = \begin{bmatrix} 1 & & & & & & \\ \omega_{21} & 1 & & & & & \\ \omega_{31} & & 1 & & & & \\ \omega_{41} & & & 1 & & & \\ \omega_{51} & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}, \qquad \Omega_i = \begin{bmatrix} I_{i-1} & & & & & \\ & 1 & & & & \\ & \omega_{i+1,i} & 1 & & & \\ & \omega_{i+2,i} & & 1 & & \\ & \omega_{i+3,i} & & & 1 & \\ & \omega_{i+4,i} & & & & 1 & \\ & & & & & & I_{n-i-4} \end{bmatrix},$$

for $i = 2, 3, \dots, n-4$. Here, I_{i-1} and I_{n-i-4} are identity matrices of order $i-1$ and $n-i-4$. For the sake of illustration, we consider the case $n = 7$ to show the transformation. From (2.1) and setting $\omega_{21} = -a/c, \omega_{31} = -d/c, \omega_{41} = -b/c$, and $\omega_{51} = -e/c$, we have

$$\Omega_1 T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a/c & 1 & 0 & 0 & 0 & 0 & 0 \\ -d/c & 0 & 1 & 0 & 0 & 0 & 0 \\ -b/c & 0 & 0 & 1 & 0 & 0 & 0 \\ -e/c & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 & 0 & e & b & d & a \\ a & c & 0 & 0 & e & b & d \\ d & a & c & 0 & 0 & e & b \\ b & d & a & c & 0 & 0 & e \\ e & b & d & a & c & 0 & 0 \\ 0 & e & b & d & a & c & 0 \\ 0 & 0 & e & b & d & a & c \end{bmatrix}$$