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Efficient weight vectors from pairwise comparison matrices

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ABSTRACT

Pairwise comparison matrices are frequently applied in multi-criteria decision making. A weight vector is called efficient if no other weight vector is at least as good in approximating the elements of the pairwise comparison matrix, and strictly better in at least one position. A weight vector is weakly efficient if the pairwise ratios cannot be improved in all non-diagonal positions. We show that the principal eigenvector is always weakly efficient, but numerical examples show that it can be inefficient. The linear programs proposed test whether a given weight vector is (weakly) efficient, and in case of (strong) inefficiency, an efficient (strongly) dominating weight vector is calculated. The proposed algorithms are implemented in Pairwise Comparison Matrix Calculator, available at pcmc.online.

1. Introduction

1.1. Pairwise comparison matrices

Pairwise comparison matrix (Saaty, 1977) has been a popular tool in multiple criteria decision making, for weighting the criteria and evaluating the alternatives with respect to every criterion. Decision makers compare two criteria or two alternatives at a time and judge which one is more important or better, and how many times. Formally, a pairwise comparison matrix is a positive matrix $\mathbf{A}$ of size $n \times n$, where $n \geq 3$ denotes the number of items to compare. Reciprocity is assumed: $a_{ij} = 1/a_{ji}$ for all $1 \leq i, j \leq n$. A pairwise comparison matrix is called consistent, if $a_{ij}a_{jk} = a_{ik}$ for all i, j, k. Let PCM_n denote the set of pairwise comparison matrices of size $n \times n$. Once the decision maker provides all the $n(n-1)/2$ comparisons, the objective is to find a weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top \in \mathbb{R}^n$ such that the pairwise ratios of the weights, w_i/w_j, are *as close as possible* to the matrix elements a_{ij}. Several methods have been suggested for this weighting problem, e.g., the eigenvector method (Saaty, 1977), the least squares method (Bozóki, 2008; Chu, Kalaba, & Spingarn, 1979; Fülöp, 2008; Jensen, 1984), the logarithmic least squares method (Crawford & Williams, 1980; 1985; de Graan, 1980), the spanning tree approach (Bozóki & Tsyganok, 2017; Lundy, Siraj, & Greco, 2017; Olenko & Tsyganok, 2016; Siraj, Mikhailov, & Keane, 2012a; 2012b; Tsyganok, 2000; 2010) besides many other proposals discussed and compared by Golany and Kress (1993), Choo and Wedley (2004), Lin (2007), Fedrizzi and Brunelli (2010). Bajwa, Choo, and Wedley (2008) not only compare seven weighting methods with respect to four criteria, but provide a detailed list of nine earlier comparative studies, too.

1.2. Weighting as a multiple objective optimization problem

The weighting problem itself can be considered as a multi-objective optimization problem which includes $n^2 - n$ objective functions, namely $\left|x_i/x_j - a_{ij}\right|$, $1 \leq i \neq j \leq n$. Let $\mathbf{A} = \left[a_{ij}\right]_{i,j=1,\ldots,n}$ be a pairwise comparison matrix and write the multi-objective optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n_{++}} \left|\frac{x_i}{x_j} - a_{ij}\right|_{1\leq i\neq j\leq n}. \tag{1}$$

Efficiency or Pareto optimality (Miettinen, 1998, Chapter 2) is a key concept in multiple objective optimization and multiple criteria decision making. See Ehrgott's historical overview (Ehrgott, 2012), beginning with Edgeworth (1881) and Pareto (1906).

Consider the functions

$$f_{ij} : \mathbb{R}^n_{++} \to \mathbb{R}, \; i, j = 1, \ldots, n,$$

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defined by

$$f_{ij}(\mathbf{x}) = \left| \frac{x_i}{x_j} - a_{ij} \right|, \;\; i, j = 1, \ldots, n, \tag{2}$$

as in Blanquero, Carrizosa, and Conde (2006, p. 273). Since $f_{ii}(\mathbf{x}) = 0$ for every $\mathbf{x} \in \mathbb{R}^n_{++}$ and $i = 1, \ldots, n$, these constant functions are irrelevant from the aspect of multi-objective optimization, so they will be simply left out from the investigations.

Let the vector-valued function $\mathbf{f} : \mathbb{R}^n_{++} \to \mathbb{R}^{n(n-1)}_{++}$ defined by its components $f_{ij}, i, j = 1, \ldots, n, \; i \neq j$. Consider the problem of minimizing $\mathbf{f}$ over a nonempty set $X \subseteq \mathbb{R}^n$ that can be written in the general form of the *vector optimization problem*

$$\min_{\mathbf{x} \in X} \mathbf{f}(\mathbf{x}). \tag{3}$$

With $X = \mathbb{R}^n_{++}$, where the latter denotes the positive orthant in $\mathbb{R}^n$, we get problem (1) in a bit more general form.

Recall the following basic concepts used for multiple objective or vector optimization. A point $\bar{\mathbf{x}} \in X$ is said to be an efficient solution of (3) if there is no $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\bar{\mathbf{x}}), \; \mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\bar{\mathbf{x}})$, meaning that $f_{ij}(\mathbf{x}) \leq f_{ij}(\bar{\mathbf{x}})$ for all $i \neq j$ with strict inequality for at least one index pair $i \neq j$. In the literature, the names Pareto-optimal, nondominated and noninferior solution are also used instead of efficient solution.

A point $\bar{\mathbf{x}} \in X$ is said to be a weakly efficient solution of (3) if there is no $\mathbf{x} \in X$ such that $\mathbf{f}(\mathbf{x}) < \mathbf{f}(\bar{\mathbf{x}})$, i.e. $f_{ij}(\mathbf{x}) < f_{ij}(\bar{\mathbf{x}})$ for all $i \neq j$. Efficient solutions are sometimes called strongly efficient.

A point $\bar{\mathbf{x}} \in X$ is said to be a locally efficient solution of (3) if there exists $\delta > 0$ such that $\bar{\mathbf{x}}$ is an efficient solution in $X \cap B(\bar{\mathbf{x}}, \delta)$, where $B(\bar{\mathbf{x}}, \delta)$ is a δ-neighborhood around $\bar{\mathbf{x}}$. The local weak efficiency is defined similarly for a point $\bar{\mathbf{x}} \in X$, the only difference is that weakly efficient solutions are considered instead of efficient solutions.

Several multi-objective optimization models have been proposed in the research of pairwise comparison matrices. Departing from Mikhailov (2006), Mikhailov and Knowles (2009) include two objective functions, the sum of least squares, written for the upper diagonal positions, and the number of minimum violations, then apply an evolutionary algorithm to generate the Pareto frontier. A third objective function, the total deviation from second-order indirect judgments, is added in Siraj, Mikhailov, and Keane (2012c).

The $n(n-1)/2$ objective functions $\left| x_i/x_j - a_{ij} \right|, \; 1 \leq i \neq j \leq n$, of the multi-objective optimization problem (1) can be aggregated into a single objective function in several ways. Their sum gives the weighting method *least absolute error* (Choo and Wedley, 2004, Section 4, LAE). If their maximum is taken into consideration, weighting method *least worst absolute error* (Choo and Wedley, 2004, Section 4, LWAE) is resulted in. The sum of their squares is the classical *least squares method* (Bozóki, 2008; Chu et al., 1979; Fülöp, 2008; Jensen, 1984). A (parametric) linear combination of the sum and the maximum is proposed by Jones and Mardle (2004) to find a compromise weight vector. A similar idea is applied in the proposal of Dopazo and Ruiz-Tagle (2011), developed for group decision problems with incomplete pairwise comparison matrices.

In the rest of the paper efficiency for problem (1), including $n(n-1)/2$ objective functions, is considered. The explicit presentation will be unavoidable for the problem specific concept of internal efficiency introduced recently in Bozóki (2014).

1.3. *Efficiency of weight vectors*

Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)^\top$ be a positive weight vector.

Definition 1.1. Weight vector $\mathbf{w}$ is called *efficient* for (1) if no positive weight vector $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^\top$ exists such that

$$\left| a_{ij} - \frac{w'_i}{w'_j} \right| \leq \left| a_{ij} - \frac{w_i}{w_j} \right| \quad \text{for all } 1 \leq i, j \leq n, \tag{4}$$

$$\left| a_{k\ell} - \frac{w'_k}{w'_\ell} \right| < \left| a_{k\ell} - \frac{w_k}{w_\ell} \right| \quad \text{for some } 1 \leq k, \ell \leq n. \tag{5}$$

Weight vector $\mathbf{w}$ is called *inefficient* for (1) if it is not efficient for (1).

If weight vector $\mathbf{w}$ is inefficient for (1) and weight vector $\mathbf{w}'$ fulfills (4)–(5), we say that $\mathbf{w}'$ *dominates* $\mathbf{w}$. Note that dominance is transitive.

It follows from the definition that an arbitrary rescaling does not influence (in)efficiency.

Remark 1. A weight vector $\mathbf{w}$ is efficient for (1) if and only if $c\mathbf{w}$ is efficient for (1), where $c > 0$ is an arbitrary scalar.

Example 1.1. Consider four criteria C_1, C_2, C_3, C_4, pairwise comparison matrix $\mathbf{A} \in PCM_4$ and its principal right eigenvector $\mathbf{w}$ as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 4 & 9 \\ 1 & 1 & 7 & 5 \\ 1/4 & 1/7 & 1 & 4 \\ 1/9 & 1/5 & 1/4 & 1 \end{pmatrix},$$

$$\mathbf{w} = \begin{pmatrix} 0.404518 \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}, \quad \mathbf{w}' = \begin{pmatrix} \mathbf{0.441126} \\ 0.436173 \\ 0.110295 \\ 0.049014 \end{pmatrix}.$$

In order to prove the inefficiency of the principal right eigenvector $\mathbf{w}$, let us increase its first coordinate: $w'_1 := 9w_4 = 0.441126$, $w'_i := w_i, i = 2, 3, 4$. The consistent approximations generated by weight vectors $\mathbf{w}$, $\mathbf{w}'$,

$$\left[\frac{w_i}{w_j} \right] = \begin{pmatrix} 1 & 0.9274 & 3.6676 & 8.2531 \\ 1.0783 & 1 & 3.9546 & 8.8989 \\ 0.2727 & 0.2529 & 1 & 2.2503 \\ 0.1212 & 0.1124 & 0.4444 & 1 \end{pmatrix}, \tag{6}$$

$$\left[\frac{w'_i}{w'_j} \right] = \begin{pmatrix} 1 & \mathbf{1.0114} & \mathbf{3.9995} & \mathbf{9} \\ \mathbf{0.9888} & 1 & 3.9546 & 8.8989 \\ \mathbf{0.2500} & 0.2529 & 1 & 2.2503 \\ \mathbf{0.1111} & 0.1124 & 0.4444 & 1 \end{pmatrix},$$

show that inequality (4) holds for all $1 \leq i, j \leq 4$, and the strict inequality (5) holds for $(k, \ell) \in \{(1, 2), (1, 3), (1, 4), (2, 1), (3, 1), (4, 1)\}$. For example, with $k = 1, \ell = 2$, $|\frac{w'_1}{w'_2} - a_{12}| = |1.0114 - 1| = 0.0114 < |\frac{w_1}{w_2} - a_{12}| = |0.9274 - 1| = 0.0726$. Weight vector $\mathbf{w}'$ dominates $\mathbf{w}$. Note that the principal right eigenvector $\mathbf{w}$ ranks the criteria as $C_2 \succ C_1 \succ C_3 \succ C_4$, while the dominating weight vector $\mathbf{w}'$ ranks them as $C_1 \succ C_2 \succ C_3 \succ C_4$.

Blanquero et al. (2006) considered the local variant of efficiency:

Definition 1.2. Weight vector $\mathbf{w}$ is called *locally efficient* for (1) if there exists a neighborhood of $\mathbf{w}$, denoted by $V(\mathbf{w})$, such that no positive weight vector $\mathbf{w}' \in V(\mathbf{w})$ fulfilling (4)–(5) exists.

Weight vector $\mathbf{w}$ is called *locally inefficient* if it is not locally efficient.

Another variant of (in)efficiency has been introduced by Bozóki (2014):