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Continuous Optimization

Variable-sized uncertainty and inverse problems in robust optimization

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ABSTRACT

In robust optimization, the general aim is to find a solution that performs well over a set of possible parameter outcomes, the so-called uncertainty set. In this paper, we assume that the uncertainty size is not fixed, and instead aim at finding a set of robust solutions that covers all possible uncertainty set outcomes. We refer to these problems as robust optimization with variable-sized uncertainty. We discuss how to construct smallest possible sets of min–max robust solutions and give bounds on their size.

A special case of this perspective is to analyze for which uncertainty sets a nominal solution ceases to be a robust solution, which amounts to an inverse robust optimization problem. We consider this problem with a min–max regret objective and present mixed-integer linear programming formulations that can be applied to construct suitable uncertainty sets.

Results on both variable-sized uncertainty and inverse problems are further supported with experimental data.

1. Introduction

Robust optimization has become a vibrant field of research with fruitful practical applications, of which several recent surveys give testimonial (see Aissi, Bazgan, & Vanderpooten, 2009; Ben-Tal, Ghaoui, & Nemirovski., 2009; Bertsimas, Brown, & Caramanis., 2011; Chassein & Goerigk., 2016b; Goerigk & Schöbel., 2016; Gorissen, Yanıkoğlu, & den Hertog., 2015). Two of the most widely used approaches to robust optimization are the so-called (absolute) min–max and min–max regret approaches (see, e.g., the classic book on the topic Kouvelis & Yu., 1997). For some combinatorial optimization problem of the form

$$\text{(P)} \qquad \min \left\{ c^t x : x \in \mathcal{X} \subseteq \{0,1\}^n \right\}$$

with a set of feasible solutions $\mathcal{X}$, let $\mathcal{U}$ denote the set of all possible scenario outcomes for the objective function parameter c. Then the min–max counterpart of the problem is given as

$$\min_{x\in\mathcal{X}} \max_{c\in\mathcal{U}} c^t x$$

and the min–max regret counterpart is given as

$$\min_{x\in\mathcal{X}} reg(x,\mathcal{U})$$

with

$$reg(x,\mathcal{U}) := \max_{c\in\mathcal{U}} (c^t x - opt(c))$$

where $opt(c)$ denotes the optimal objective value for problem (P) with objective c.

In the recent literature, the problem of finding suitable sets $\mathcal{U}$ has come to the focus of attention, see Bertsimas and Sim. (2004), Bertsimas, Pachamanova, and Sim. (2004) and Bertsimas and Brown. (2009). This acknowledges that the set $\mathcal{U}$ might not be "given" by a real-world practitioner, but is part of the responsibility of the operations researcher.

In this paper we consider the question how robust solutions change when the size of the uncertainty set changes. We call this approach variable-sized robust optimization and analyze how to find minimal sets of robust solutions that can be applied to any possible uncertainty sets. This way, the decision maker is presented with candidate solutions that are robust for different-sized uncertainty sets, and he can choose which suits him best. Results on this approach for min–max robust optimization are discussed in Section 2.

The notion of *variable uncertainty* has also been used in Poss. (2013), but is different to our approach: In their paper, the size of the uncertainty set depends on the solution x, i.e., $\mathcal{U} = \mathcal{U}(x)$, while we use size parameter that does not depend on x. Further related is the notion of parametric programming

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(see, e.g., Carstensen, 1983). Our approach can be seen as being a parametric robust optimization problem.

As a special case of variable-sized robust optimization, we consider the following question: Given only a nominal problem (P) with objective $\hat{c}$, how large can an uncertainty set become, such that the nominal solution still remains optimal for the resulting robust problem? Due to the similarity in our question to inverse optimization problems, see, e.g., Ahuja and Orlin. (2001), Heuberger (2004), Alizadeh, Burkard, and Pferschy (2009) and Nguyen and Chassein. (2015) we denote this as the inverse perspective to robust optimization. The approach from Carrizosa and Nickel (2003) is remotely related to our perspective. There, the authors define the robustness of a solution via the largest possible deviation of the problem coefficients in a facility location setting. In a similar spirit, the *R-model* for robust satisficing (Jaillet, Jena, Ng, & Sim., 2016) aims at finding a solution that remains feasible for an uncertainty set that is as large as possible. Our approach can also be used as a means of sensitivity analysis. Given several solutions that are optimal for some nominal problem, the decision maker can choose one that is most robust in our sense. In the same vein, one can check which parts of a solution are particularly fragile, and strengthen them further. This approach is presented for min–max regret in Section 3.

Our paper closes with a conclusion and discussion of further research directions in Section 4.

2. Variable-sized min–max robust optimization

In this section, we analyze how optimal robust solutions change when the size of the uncertainty set increases. We assume to have information about the midpoint (nominal) scenario $\hat{c}$, and the shape of the uncertainty set $\mathcal{U}$. The actual size of the uncertainty set is assumed to be uncertain.

More formally, we assume that the uncertainty set is given in the form

$$\mathcal{U}_\lambda = \{\hat{c}\} + \lambda B$$

where B is a convex set containing the origin and $\hat{c} > 0$ is the midpoint of the uncertainty set. The parameter $\lambda \geq 0$ is an indicator for the size of the uncertainty set. For $\lambda = 0$, we have $\mathcal{U}_0 = \{\hat{c}\}$, i.e., the nominal problem, and for $\lambda \to \infty$ we obtain the extreme case of complete uncertainty.

We consider the min–max robust optimization problem

$$\min_{x\in\mathcal{X}} \max_{c\in\mathcal{U}_\lambda} c^t x. \tag{P(λ)}$$

The goal of variable-sized robust optimization is to compute a minimal set $\mathcal{S} \subset \mathcal{X}$ such that for any $\lambda \geq 0$, $\mathcal{S}$ contains a solution that is optimal for P(λ). Note that for $\lambda = 0$, set $\mathcal{S}$ must contain a solution $\hat{x}$ that is optimal for the nominal problem. By increasing λ, we trace how this solution needs to change with increasing degree of uncertainty.

Section 2 is structured as follows. In Section 2.1, we demonstrate the close relation between the variable-sized robust problem and a bicriteria optimization problem. We consider different shapes for set B in Section 2.2 and discuss the resulting variable-sized robust optimization problems. In Section 2.3, we transfer the general results of the previous section to the shortest path problem. We end the discussion of the shortest path problem with a case study.

2.1. Relation to bicriteria optimization

We reformulate the objective function of P(λ):

$$\max_{c\in\mathcal{U}_\lambda} c^t x = \hat{c}^t x + \max_{c\in\lambda B} c^t x = \hat{c}^t x + \max_{\tilde{c}\in B} \lambda \tilde{c}^t x$$

$$= \hat{c}^t x + \lambda \max_{\tilde{c}\in B} \tilde{c}^t x = f_1(x) + \lambda f_2(x)$$

where $f_1(x) = \hat{c}^t x$ and $f_2(x) = \max_{c\in B} c^t x$. Note that both functions are convex. It is immediate that the variable-sized robust optimization problem is closely related to the bicriteria optimization problem:

$$\min_{x\in\mathcal{X}} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

We define the map $F : \mathcal{X} \to \mathbb{R}^2_+, F(x) = (f_1(x), f_2(x))^t$ which maps every feasible solution in the objective space. Further, we define the polytope $\mathcal{V} = \text{conv}(\{F(x) : x \in \mathcal{X}\}) + \mathbb{R}^2_+$. We call a solution $x \in \mathcal{X}$ an *efficient extreme solution* if $F(x)$ is a vertex of $\mathcal{V}$. Denote the set of all efficient extreme solutions with $\mathcal{E}$. We call two different solutions $x \neq x'$ *equivalent* if $F(x) = F(x')$. Let $\mathcal{E}_{\min} \subset \mathcal{E}$ be a maximal subset such that no two solutions of $\mathcal{E}_{\min}$ are equivalent. The next lemma gives the direct relation between $\mathcal{E}_{\min}$ and the variable-sized robust optimization problem.

Lemma 1. *$\mathcal{E}_{\min}$ is a solution of the variable sized robust optimization problem.*

Proof. We need to prove two properties:

(I) For every $\lambda \geq 0$ there exists a solution in $\mathcal{E}_{\min}$ which is optimal for P(λ).
(II) $\mathcal{E}_{\min}$ is a smallest possible set with property (I).

(I) Let $\lambda \geq 0$ be fixed. We transfer the problem P(λ) in the objective space. The optimal value of P(λ) is equal to the optimal value of problem O(λ) since each optimal solution of this problem is a vertex of $\mathcal{V}$.

$$\min_{v\in\mathcal{V}} v_1 + \lambda v_2 \tag{O(λ)}$$

Let v^* be the optimal solution of O(λ). By definition, $\mathcal{E}_{\min}$ contains a solution x^* with $F(x) = v^*$, i.e., an optimal solution for P(λ).

(II) Note that there is a one-to-one correspondence between vertices of $\mathcal{V}$ and solutions in $\mathcal{E}_{\min}$. Since for each vertex v' of $\mathcal{V}$ a $\lambda' \geq 0$ exists such that v' is optimal for O(λ), it follows that $\mathcal{E}_{\min}$ is indeed minimal. Note that it is important to ensure that $\mathcal{E}_{\min}$ contains no equivalent solutions. □

Different methods are known to compute $\mathcal{E}_{\min}$. The complexity of these methods depends linearly on the size of $\mathcal{E}_{\min}$. To find an additional efficient extreme solution a weighted sum problem of the form $\min_{x\in\mathcal{X}} f_1(x) + \lambda f_2(x)$ needs to be solved. In the following, we denote by T the time which is necessary to solve the nominal problem $\min_{x\in\mathcal{X}} c^T x$ and by T' the time that is necessary to solve a weighted sum problem. Note that if $f_2(x)$ is linear in x, $O(T') = O(T)$. We restate the following well known result (see e.g. Mordechai, 1986).

Lemma 2. *$\mathcal{E}_{\min}$ can be computed in $O(|\mathcal{E}_{\min}|T')$.*

Using Lemma 1, we can transfer this result to the variable-sized robust problem.

Theorem 3. *The variable-sized robust problem can be solved in $O(|\mathcal{E}_{\min}|T')$.*

2.2. General results

We assume that B is the unit ball of some norm. An overview of different norms and the corresponding functions $\max_{c\in\mathcal{U}_\lambda} c^t x$ is presented in Table 1. This list does not cover the large set of different uncertainty sets which is studied in the literature. For more