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Continuous Optimization

Benders decomposition and column-and-row generation for solving large-scale linear programs with column-dependent-rows

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ABSTRACT

In a recent work, Muter, Birbil, and Bülbül, (2013) identified and characterized a general class of linear programming (LP) problems – known as problems with column-dependent-rows (CDR-problems). These LPs feature two sets of constraints with mutually exclusive groups of variables in addition to a set of structural linking constraints, in which variables from both groups appear together. In a typical CDR-problem, the number of linking constraints grows very quickly with the number of variables, which motivates generating both columns and their associated linking constraints simultaneously on-the-fly. In this paper, we expose the decomposable structure of CDR-problems via Benders decomposition. However, this approach brings on its own theoretical challenges. One group of variables is generated in the Benders master problem, while the generation of the linking constraints is relegated to the Benders subproblem along with the second group of variables. A fallout of this separation is that only a partial description of the dual of the Benders subproblem is available over the course of the algorithm. We demonstrate how the pricing subproblem for the column generation applied to the Benders master problem does also update the dual polyhedron and the existing Benders cuts in the master problem to ensure convergence. Ultimately, a novel integration of Benders cut generation and the simultaneous generation of columns and constraints yields a brand-new algorithm for solving large-scale CDR-problems. We illustrate the application of the proposed method on a time-constrained routing problem. Our numerical experiments confirm the outstanding performance of the new decomposition method.

1. Introduction

We have recently developed a simultaneous column-and-row generation methodology (Muter, Birbil, & Bülbül, 2013a) for a general class of linear programming problems. The. problems that belong to this class have an interesting structure In their most general form, they feature two sets of constraints with mutually exclusive groups of variables in addition to a set of *linking* constraints, in which variables from both groups appear together. The list of applications which fit into this framework and satisfy the assumptions of our analysis include multi-stage cutting stock (Zak, 2002), P-median facility location (Avella, Sassano, & Vasilev, 2007), multi-commodity capaci tated network design (Frangioni & Gendron, 2009; Katayama, Chen, & Kubo, 2009), two-stage batch scheduling (Wang & Tang, 2010), robust crew pairing (Muter et al., 2013b), and time-constrained routing (Avella, D'Auria, & Salerno, 2006; Muter, Birbil, Bülbül, & Şahin, 2012). These problems are frequently formulated with too many variables to be included explicitly in the model at the outset and are therefore typically attacked by column generation techniques (Dantzig & Wolfe, 1960; Lübbecke & Desrosiers, 2005). The additional challenge here is that the number of linking constraints is either too large which precludes us from incorporating these constraints directly in the formulation, or an explicit description of the full set of linking constraints is only available in the presence of the entire set of variables. Therefore, whenever these problems are solved by column generation, the introduction of new columns leads to the generation of new linking constraints. That is, the sequence of LPs solved during column generation does not only grow column-wise but also row-wise through the addition of new linking constraints. Consequently, we refer to this class of formulations as problems with *column-dependent-rows*, or more concisely as CDR-problems. A key point here is that these new linking constraints are structural constraints required for the validity of the formulation. This is a defining property which clearly distinguishes our

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work from the branch-and-cut-and-price setting as we elaborate upon in the next paragraph. The primary challenge in solving CDR-problems via column generation is to price out the columns absent from the formulation correctly as the dual variables associated with the missing linking constraints are unknown. To overcome this difficulty, we have developed a *thinking-ahead approach*, which computes the correct values of the dual variables of the missing linking constraints as if they were present in the formulation (Muter et al., 2013a). Recently, simultaneous generation of columns and rows has also been studied in other settings by Frangioni and Gendron (2013) and Sadykov and Vanderbeck (2013).

A large number of problems solved by column generation originally involve integrality constraints for some or all of the variables. To solve such problems to optimality, column generation is generally integrated into the branch-and-bound method giving rise to the branch-and-price method (Barnhart, Johnson, Nemhauser, Savelsbergh, & Vance, 1998). Generating strengthened bounds at the nodes of the branch-and-price tree can be achieved through valid inequalities obtained by solving a separation subproblem. This use of column and row generation together in a branch-and-bound setting is known as branch-and-cut-and-price; see the works of Desaulniers, Desrosiers, and Spoorendonk (2011) and Desrosiers and Lübbecke (2011). In the branch-and-cut-and-price framework, rows and columns are generated sequentially and independently from each other by solving the separation and pricing subproblems, respectively. This is fundamentally different from the setting in Muter et al. (2013a) and in this paper, which requires us to generate both columns and their associated linking constraints interdependently on-the-fly.

In this paper, we extend our previous work on CDR-problems and solve them by using Benders decomposition (Benders, 1962). This decomposition technique partitions the variables into two smaller problems –the Benders master problem and the Benders subproblem– so that the overall problem can be handled efficiently with an iterative algorithm known as delayed cut generation. Rahmaniani, Crainic, Gendreau, and Rei (2017) provides a comprehensive survey of applications solved by Benders decomposition, including capacitated facility location (Fischetti, Ljubić, & Sinnl, 2016), production routing (Adulyasak, Cordeau, & Jans, 2015), multi-period hub location (Gelareh, Monemi, & Nickel, 2015), and strip packing (Côté, Dell'Amico, & Iori, 2014). As we shall discuss in the next section, the structure of CDR-problems qualifies for such a decomposition. However, we observe that a direct application of Benders decomposition is not possible because we do not have the complete description of the dual of the Benders subproblem – the dual slave problem – during the iterations. In particular, the dimension of the feasible region of the dual slave problem – the dual polyhedron – increases as new linking constraints are introduced. This novel structure leads us to reconsider the fundamental parts of Benders decomposition; solving a sequence of relaxed Benders master problems and applying delayed cut generation by solving the dual slave problem in between. The proposed analysis along with our observations constitute the main contributions of this work: We develop a new Benders decomposition methodology for solving large-scale linear programs with column-dependent-rows. This approach induces a novel integration of the delayed (Benders) cut generation and simultaneous column-and-row generation for solving large-scale CDR-problems. To illustrate the application of the proposed method, we explain each step on the time-constrained routing problem. Our numerical experiments confirm that the new decomposition method outperforms not only the off-the-shelf solvers but also our previous algorithm for CDR-problems presented in Muter et al. (2013a).

2. Motivation

We devote this section to explaining our motivation for developing a new Benders decomposition methodology for solving CDR-problems. To this end, we first revisit the generic mathematical model for CDR-problems:

$$(\mathbf{MP}) \qquad \text{minimize} \quad \sum_{k\in K} c_k y_k + \sum_{n\in N} d_n x_n,$$

$$\text{subject to} \quad \sum_{k\in K} A_{jk} y_k \geq a_j, \quad j\in J, \tag{MP-y}$$

$$\sum_{n\in N} B_{mn} x_n \geq b_m, \quad m\in M, \tag{MP-x}$$

$$\sum_{k\in K} C_{ik} y_k + \sum_{n\in N} D_{in} x_n \geq r_i, \quad i\in I(K,N), \tag{MP-yx}$$

$y_k \geq 0,\, k\in K, x_n \geq 0,\, n\in N.$

In general, we allow for exponentially many y- and x- variables in the *master problem* formulation (**MP**) above and therefore reckon that any generic viable algorithm for solving CDR-problems must be able to generate both types of variables dynamically in a column generation framework. The cardinality of the constraints (MP-y) and (MP-x) is polynomially bounded in the size of the problem, and these are directly incorporated into the model. However, the cardinality of the set of linking constraints (MP-yx) depends on $|K|$ and $|N|$ and is either theoretically or practically too large – see Example 3.1 in Section 3. Note that the dependence of the linking constraints on the variables is conveyed through the notation $I(K,N)$. This non-standard structure of (**MP**) prompts us to search for alternatives to handle the linking constraints (MP-yx) in a column generation scheme.

A key observation that motivates the solution approach for CDR-problems in this paper is that the constraints (MP-y) and (MP-x) impose conditions only on the y- and x-variables, respectively. Therefore, these two groups of variables can be handled in two separate problems. This is a typical structure amenable to Benders decomposition. In particular, we project out the x-variables and relegate them to the subproblem while the y-variables are kept in the master problem. This choice is not arbitrary; the underlying reason will be clear in Section 3.1, where we qualify the y-variables as the *primary* set of variables in some sense. Formally, we write

$$\begin{aligned}
\text{minimize} \quad & \sum_{k\in K} c_k y_k + z(\boldsymbol{y}), && \\
\text{subject to} \quad & \sum_{k\in K} A_{jk} y_k \geq a_j, && j\in J, \\
& y_k \geq 0, && k\in K,
\end{aligned} \tag{1}$$

where for a fixed $\bar{\boldsymbol{y}}$ we have

$$\begin{aligned}
(\mathbf{BSP}) \quad z(\bar{\boldsymbol{y}}) = \quad & \text{minimize} \quad \sum_{n\in N} d_n x_n, && \\
(v_m) \quad & \text{subject to} \quad \sum_{n\in N} B_{mn} x_n \geq b_m, && m\in M, \\
(w_i) \quad & \sum_{n\in N} D_{in} x_n \geq r_i - \sum_{k\in K} C_{ik}\bar{y}_k, && i\in I(K,N), \\
& x_n \geq 0, && n\in N.
\end{aligned} \tag{2}$$

This problem is referred to as the *Benders subproblem*, where the dual variables are indicated in parentheses to the left of their respective constraints. Using LP duality, we obtain the equivalent