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Continuous Optimization

Lanchester model for three-way combat

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ABSTRACT

Lanchester (1960) modeled combat situations between two opponents, where mutual attrition occurs continuously in time, by a pair of simple ordinary (linear) differential equations. The aim of the present paper is to extend the model to a conflict consisting of three parties. In particular, Lanchester's main result, i.e. his Square Law, is adapted to a triple fight. However, here a central factor – besides the initial strengths of the forces – determining the long run outcome is the allocation of each opponent's efforts between the other two parties. Depending on initial strengths, (the) solution paths are calculated and visualized in appropriate phase portraits. We are able identify regions in the state space where, independent of the force allocation of the opponents, always the same combatant wins, regions, where a combatant can win if its force allocation is wisely chosen, and regions where a combatant cannot win itself but determine the winner by its forces allocation. As such, the present model can be seen as a forerunner of a dynamic game between three opponents.

1. Introduction

Lanchester (1916) applied a pair of linear ordinary differential equations to understand the dynamics of a battle between two opponents. He was inspired by the attrition and exhaustion of fighters in air combats in World War I. Since then many papers have been published on that and related issues, see, e.g. Morse and Kimball (1951); see also Washburn and Kress (2009), Kress (2012). It is surprising, however, that while Lanchester attrition duels are prevalent in the literature, there are hardly any models for combat situations involving three sides or more sides. An exception is Syms and Solymar (2017), who analyze the Lanchester model with area fire and recruitment. Furthermore, Lin and MacKay (2014) studied the optimal policy for a one-against-many combat in a Lanchester framework. The aim of the present paper is to extend the Lanchester model to three combatants and analyze the more general case where all of the opponents are engaged in combat against each other in a Lanchester framework with aimed fire and without recruitment (which is difficult in an ongoing insurgency).

In the classic Lanchester model two opponents fight each other. Their sizes are considered as state variables. The decrease of their forces over time depends on the size of the forces and their per capita effectiveness measured by their respective attrition rates. There are two main types of Lanchester models corresponding to direct and area fire. The direct fire model results in a quadratic equation (conserved quantity) that is manifested in the Square Law. The area fire model induces a linear state equation and, accordingly, is governed by the Linear Law. Although there exist stochastic versions of the models (e.g., Kress & Talmor, 1999) the commonly used models are deterministic. Deitchman (1962) combined the two types of Lanchester models and defined the "Guerrilla Warfare" model where one side (the guerrillas) utilize direct fire, while the other side (regular forces) use area fire.

Lanchester models are purely attritional and ignore the crucial role of situational awareness and intelligence. Attempts to generalize Lanchester theory by incorporating the effect of information are reported in Kress and Szechtman (2009), Kaplan, Kress, and Szechtman (2010). Schramm and Gaver (2013) combine the Lanchester model with a deterministic epidemic model to account for the impact of information spreading.

Over the years there have been many analyses and extensions of Lanchester models. For example, Bracken (1995) validates the linear Lanchester model with historical data from World War II. Chen and Chu (2001) extend this approach by taking into

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account the timing of a shift between defense and attack. Stochastic aspects in Lanchester models are discussed in Hausken and Moxnes (2000, 2002, 2005). Zero-sum attrition games on a network which differ with respect to the information structure are analyzed in Hohzaki and Higashio (2016). MacKay (2015) combines the Richardson, Lanchester and Deitchman model to find that the typical outcome of such a combined model is not the annihilation of one opponent but a stale-mate, where both parties remain active forever in a steady state.

Of course, there are also many interesting papers outside the Lanchester framework which analyze important aspects of combats such as Peng, Zhai, and Levitin (2016), who analyze a game between an attacker and a defender under the deployment of false targets, and Zhai, Ye, Peng, and Wang (2017), who consider the issue of infrastructure protection within a two-player zero-sum game. See Hausken and Levitin (2012) for a review of papers related to the defense and attack of systems.

The aim of the present paper is to extend the analysis of the classic Lanchester model of direct fire to a three-sided battle. There are many recent and historic examples for three-sided combats such as the Bosnian Civil War and the Iraq Civil War (government vs. Shias vs. Sunnis), see Syms and Solymar (2017) for more examples. The analysis in this paper is motivated by recent events in Syria, where at least six active parties (not to mention the "big" players such as Russia, Turkey and Iran) – al-Qaeda affiliated groups (e.g., Jabahat al Nusra), ISIS, the free Syrian army, Hezbollah, Kurds and Assad regime forces – fight each other to gain control on land, people and national assets. In contrast to a one-on-one engagement, additional parameters are needed to indicate how each side's firepower should be allocated between its two opponents. Compare also the literature on optimal fire distribution where one of the two opponents consists of two heterogeneous forces, see e.g. Taylor (1974), Lin and MacKay (2014). We assume that each party commits to allocate a fixed percentage of its efforts toward each opponent throughout the conflict, e.g., one-third directed against enemy 1 and two-thirds against enemy 2. We will show how the initial force-size of the three opponents together with the attrition rates and the fire-allocation tactics determine the winner of the battle. More complicated, dynamically adjusting strategies are possible in principle, but the fixed proportions problem is interesting in and of itself.

We use eigenvalue analysis to identify surfaces separating regions of initial states that differ in the way the conflict is played out. By restricting the state space to the unit simplex we obtain an illustrative description of the solution paths. Moreover, we are able to identify in that simplex, for each side, its winning regions – initial conditions that guarantee its win.

The paper is organized as follows. In Section 2 we present the model and characterize the solution. In Section 3 we discuss the numerical solution of the problem. Section 4 concludes.

2. Lanchester model with three combatants

We formulate a two-stage Lanchester model in Section 2.1, and introduce some important concepts in Section 2.2. We recapitulate the important properties of the Lanchester model with two sides in Section 2.3, and derive the corresponding properties for the model with three sides in Section 2.4.

2.1. Two-stage model

We consider a situation where each force among three is engaged in combat against the other two (henceforth called also *sides* or *combatants*). The strength of each of the forces at time t is denoted as $I_j(t),\ \ j=0,1,2$. In fact the strength of the forces $I_j,\ \ j=0,1,2$ are normalized by the initial total size $N=\sum_{j=0}^{2} I_j(0)$ and hence denote the relative strengths. Due to the linearity of the ODEs the total strength is given by the multiplication with N. The battle comprises two stages. It is not possible to fight both opponents simultaneously with same forces, therefore it is necessary that each side splits its forces between the two opponents in the first stage of the battle. The fraction of the force of side j that is allocated to engage side i is denoted by the parameter $y_{ij},\ \ i,j=0,1,2$. It is assumed that the opponents do not adapt the allocation of their forces over time. They also cannot form coalitions. The parameters $a_{i,j}$ denote the attrition rates when j engages i with $i,j=0,1,2$.

If one of the three forces is annihilated, the two remaining sides continue in a Square Law battle in which all of their forces are engaged. Formally,[1]

$$\dot{I}_0(t) = -a_{01}y_{01}I_1(t) - a_{02}y_{02}I_2(t), \quad t \in [0,\tau_1\rangle \tag{1a}$$

$$\dot{I}_1(t) = -a_{10}y_{10}I_0(t) - a_{12}y_{12}I_2(t), \quad t \in [0,\tau_1\rangle \tag{1b}$$

$$\dot{I}_2(t) = -a_{20}y_{20}I_0(t) - a_{21}y_{21}I_1(t), \quad t \in [0,\tau_1\rangle \tag{1c}$$

where τ_1 is the time when the first force among the three is annihilated. The initial sizes of the forces are given by

$$I_j(0) = I_j^0 \geq 0,\ \ j=0,1,2, \quad \text{and} \quad \sum_{j=0}^{2} I_j^0 = 1. \tag{1d}$$

If the forces of the remaining sides k, l with $k \neq l$ are strictly positive at τ_1, then at the second stage

$$\dot{I}_k(t) = -a_{kl}I_l(t), \quad t \in [\tau_1,\tau_2\rangle \tag{1e}$$

$$\dot{I}_l(t) = -a_{lk}I_k(t), \quad t \in [\tau_1,\tau_2\rangle \tag{1f}$$

$$\dot{I}_j(t) = 0,\ j = 3-(k+l), \quad t \in [\tau_1,\tau_2\rangle \tag{1g}$$

where τ_2 is the time when the second stage ends where at least one of the two remaining sides from stage one is annihilated too.

The coefficients in the first stage satisfy

$$0 \leq y_{ij} \leq 1, \quad \sum_{i \neq j} y_{ij} = 1, \quad a_{ij} > 0, \quad i,j=0,1,2,\ \ i \neq j, \tag{1h}$$

and

$$[\tau_1,\tau_2\rangle := \begin{cases} [\tau_1,\tau_2] & \text{if} \tau_2 < \infty \\ [\tau_1,\infty) & \text{if} \tau_2 = \infty. \end{cases}$$

The restriction as in the first stage Eq. (1d) is the normalization mentioned before that allows us to consider the unit tetrahedron as phase space with the initial states (force sizes) lying in the unit 2-simplex, subsequently denoted as Δ.

For the second stage we assume that the combat attrition rates remain the same as in the first stage.

2.2. Extinction times and curves

The next sections address the problem of classifying possible scenarios for the solutions of Eq. (1). Specifically we are interested in determining the first and second extinction times τ_1 and τ_2 and if there exists an opponent $I_k(\cdot)$ who wins in the sense that $I_k(\tau_2) > 0$. Thus, we give the following definitions (Table 1).

Definition 1 (Extinction times, survivors, winner and stages)**.** Let $I(\cdot) = (I_0(\cdot), I_1(\cdot), I_2(\cdot))^\top$ be the solution of Eqs. (1a)–(1d). The time

[1] As usual, the dot denotation refers to the time derivative, i.e. $\dot{I}_i = \frac{dI_i}{dt}$, $i=0,1,2$.