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Innovative Applications of O.R.

Analysis of reliability systems via Gini-type index

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ABSTRACT

Different strategies of reliability theory for the analysis of coherent systems have been studied by various researchers. Here, the Gini-type index is utilized as an applicable tool for the study and comparison of the ageing properties of complex systems. A new stochastic order in terms of Gini-type index is introduced to compare the speed of ageing of components and systems. The parallel-series and series-parallel systems with shared components are studied by their corresponding Gini-type indexes. Also, the generalization of Gini-type index for the multidimensional case is discussed, and is used to compare components lifetimes properties in the presence of other dependent components. It is shown that the ageing properties of a component lifetime can differ when the other components are working or have already failed. Numerous illustrative examples are given for better intuition of Gini-type and generalized Gini-type indexes throughout the paper.

1. Introduction

Optimizing the system lifetime is a relevant problem in reliability theory, and leads to interesting questions in mathematical statistics and probability modelling. Many investigations have been oriented to the development of optimization strategies under various assumptions, especially when the system components are assumed to be dependent. Spizzichino (2001) studied notions of dependence and notions of ageing, which provide the tools to obtain inequalities for conditional survival probabilities. Navarro, Ruiz and Sandoval (2005) studied some comparisons between coherent systems with dependent components. Khaledi and Shaked (2007) stochastically compared the residual lifetimes of coherent systems with identical or different types of components. Navarro, del Águila, Sordo, and Suárez Llorens (2013) obtained ordering properties for coherent systems with possibly dependent identically distributed components. Their results are based on a representation of the system reliability function as a distorted function of the common component reliability function.

Recently, Navarro, Pellerey, and Di Crescenzo (2015) considered a general coherent system with independent or dependent components, and assumed that the components are randomly chosen from two different stocks. They provided sufficient conditions on the components lifetimes and on the random number of components chosen from the two stocks in order to improve the reliability of the whole system. See, also, Navarro and Spizzichino (2010) and Di Crescenzo and Pellerey (2011) for the analysis and the comparison of parallel and series systems with heterogeneous components sharing the same copula, or with components linked via suitable mixtures.

The stochastic order-based approach has also been exploited by Gupta, Misra, and Kumar (2015), aiming to compare the residual lifetime and the inactivity time of a used coherent system with the lifetime of the similar coherent system composed by used components. A new notion for the comparison of the hazard rates of random lifetimes has been introduced by Belzunce, Martinez-Riquelme, Pellerey, and Zalzadeh (2016), where the mutual dependence is taken into account.

Borgonovo, Aliee, Glass, and Teich (2016) studied modern digital systems which may exhibit a non-coherent behaviour and measured the importance of system components. They also proposed a new importance measure for time-independent reliability analysis.

More recently, Navarro, Longobardi, and Pellerey (2017) provide a general procedure based on the recent concept of generalized distorted distributions to get representations for the reliability functions of inactivity times of coherent systems with dependent components, by which one can compare systems inactivity times.

Here, we propose to adopt new applicable tools to gain information on the ageing characteristics of reliability systems, based on the Gini-type (GT) index defined by Kaminskiy and Krivtsov (2010). Such index is expressed in terms of the cumulative hazard rate function of a random lifetime. Its definition recalls the well-known ‘Gini coefficient’, which is largely used in the economics

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literature to analyse incoming distributions. In the context of system reliability, the GT index is a measure of the ageing property of a random lifetime. As a consequence, in the analysis of point process describing the occurrence of system failure times, the GT index is useful to determine if the system is stable, or is improving, or is deteriorating. Our aim is to investigate various properties of that index, and to define a new proper stochastic order based on it, which is helpful to assess the ageing properties of random lifetimes. Specific applications, which involve comparisons and ageing properties of parallel-series and series-parallel systems with shared components, are then provided.

Some extensions of GT index to multidimensional case are also thoroughly investigated, by which the multivariate and conditional ageing properties of the lifetime variables are accessible. It is worth pointing out that our approach allows to study components lifetimes properties in the presence of other sharing dependent components. Specifically, we show that the multidimensional GT index is able to describe how the ageing properties of a component lifetime can vary when the other (dependent) components are working or have already failed.

This paper is organized as follows. In Section 2 the GT index is introduced and its application in reliability theory is expressed. A characterization result of the Weibull distribution in terms of the GT index is also provided. In Section 3 we define the preannounced stochastic order in terms of GT index and discuss its properties. Section 4 is devoted to application of the GT index to series systems. In Sections 5 and 6 two structures for complex systems with shared components are introduced. Their ageing properties are studied and compared by means of the GT index.

In Section 7 we define the generalized GT index for bivariate and multivariate random lifetimes of the components working in the same environment. Furthermore, a new stochastic order is defined in terms of the generalized GT index and of suitable cumulative hazards. Such stochastic order is useful to compare the lifetimes of components in the presence of other dependent components and under various operational conditions.

Finally, in Section 8 the vector GT index is introduced for non-negative random variables, based on the multivariate failure rate of multiple components of a system.

Note that throughout this paper, we say that X is a random lifetime in order to refer to a non-negative absolutely continuous (a.c.) random variable with continuous density function (d.f.). Moreover, 'log' means natural logarithm, and prime denotes derivative.

2. Gini-type index

The ageing behaviour of repairable or non-repairable systems is vitally important for maintenance strategies. Kaminskiy and Krivtsov (2010) introduced a simple index which helps to assess the degree of ageing, or rejuvenating, of repairable (or non-repairable) systems. Let X be a non-negative random lifetime of a component or a system. For $t \geq 0$, let

$$\bar{F}(t) = \mathbb{P}(X > t)$$

and

$$H_x(t) = -\log \bar{F}(t) \tag{1}$$

represent its survival function and cumulative hazard rate function, respectively. Assuming that

$$D_X^1 := \{t > 0 : 0 < \bar{F}(t) < 1\}, \tag{2}$$

the GT index is introduced for all $t \in D_X^1$ as follows (see Kaminskiy & Krivtsov, 2010). We recall that $h_X(t) = \frac{d}{dt} H_X(t)$, $t \in D_X^1$, is the hazard rate of X.

Definition 1. The GT index for a random lifetime X, in time interval $(0, t]$, is

$$\mathrm{GT}_x(t) = 1 - \frac{2}{t\, H_x(t)} \int_0^t H_x(u)\, \mathrm{d}u, \quad t \in D_x^1. \tag{3}$$

It is shown that GT index satisfies the inequality

$$-1 < \mathrm{GT}_x(t) < 1 \quad \text{for all } t \in D_X^1.$$

Let us now recall some well-known ageing notions that will be related to the properties of GT index.

Definition 2. A random lifetime X is said to be

- IFR (increasing failure rate) if $h_X(t)$ is non-decreasing $\forall t$,
- IFRA (increasing failure rate average) if $H_X(t)/t$ is non-decreasing $\forall t$. Dually, X is said to be
- DFR (decreasing failure rate) if $h_X(t)$ is non-increasing $\forall t$,
- DFRA (decreasing failure rate average) if $H_X(t)/t$ is non-increasing $\forall t$.

For more details on the aforementioned notions, see Shaked and Shanthikumar (2007) or Barlow and Proschan (1996).

We point out that the GT index can be assumed as a measure of the ageing property of the underlying random lifetime. Indeed, since $H_X(0) = 0$ and H_X is an a.c. function, the following result holds (see Kaminskiy & Krivtsov, 2010).

Proposition 1. *For a random lifetime X we have that*

(i) *$GT_X(t) \geq (\leq)\ 0$ for all $t \in D_X^1$ if and only if X is IFR (DFR);*
(ii) *$GT_X(t) = 0$ for all $t \in D_X^1$ if and only if X is CFR (constant failure rate), i.e. X has exponential distribution.*

Clearly, the GT index changes its sign when the hazard rate is non-monotonic. For instance, if $h_X(t) = t(t-1)^2$, $t \geq 0$, then $\mathrm{GT}_X(t) = \frac{3}{5} + \frac{4(t-2)}{5(6+t(3t-8))}$, $t \geq 0$, which is first positive, then negative, and finally positive as t increases.

According to Kaminskiy and Krivtsov (2010), it should be mentioned that the GT index is, in a sense, distribution-free. Moreover, the index introduced in Definition 1 is defined similarly as the 'Gini coefficient', which is used in macroeconomics for analysing income distributions. In the reliability analysis of repairable systems, it is highly interesting to distinguish if the point process of the failure times is close to, or far from, the homogeneous Poisson process. The analysis of the GT index is thus useful to determine if the system is stable, or is improving, or is deteriorating. Indeed, if X describes the consecutive failure times in a repairable system, the condition that $\mathrm{GT}_X(t)$ is positive (negative) expresses that the system is deteriorating (improving), whereas a $\mathrm{GT}_X(t)$ vanishing means that X has exponential distribution, i.e. the system is a homogeneous Poisson process (see Kaminskiy and Krivtsov, 2010 for further details).

Let us now investigate if the GT index can be a constant value for further cases rather than the exponential distribution. This provides us a characterization result of the Weibull distribution in terms of the GT index, which extends case (ii) of Proposition 1.

Theorem 1. *The random lifetime X has Weibull distribution if and only if the corresponding* GT *index is constant.*

Proof. The proof of one side is straightforward. Thus, suppose that GT index of X is constant, i.e. $\mathrm{GT}_X(t) = r$ for $t \in D_X^1$, with $-1 < r < 1$. Hence, from (3) we have

$$\frac{1}{t\, H_x(t)} \int_0^t H_x(u)\, \mathrm{d}u = \frac{1-r}{2}, \quad t \in D_x^1.$$

Differentiating both sides with respect to t, since $H_X(t)$ is differentiable, we obtain

$$H_x'(t) - \frac{1+r}{1-r}\frac{H_x(t)}{t} = 0. \tag{4}$$