ARTICLE IN PRESS
JID: EOR [m5G;April 4, 2017;20:58]

European Journal of Operational Research 000 (2017) 1–10

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

Higher-degree stochastic dominance optimality and efficiency

Yi Fang[a,*], Thierry Post[b]

[a] Center for Quantitative Economics, Jilin Universityand Business school, Jilin University, Changchun 130012, China
[b] Thierry Post is Professor of Finance at the Graduate School of Business of Nazarbayev University, Astana 010000, Kazakhstan

ARTICLE INFO

Article history:
Received 15 March 2016
Accepted 10 March 2017
Available online xxx

Keywords:
Decision analysis
Stochastic dominance
Expected utility
Linear programming
Convex quadratic programming

ABSTRACT

We characterize a range of Stochastic Dominance (SD) relations by means of finite systems of convex inequalities. For 'SD optimality' of degree 1 to 4 and 'SD efficiency' of degree 2 to 5, we obtain exact systems that can be implemented using Linear Programming or Convex Quadratic Programming. For SD optimality of degree five and higher, and SD efficiency of degree six and higher, we obtain necessary conditions. We use separate model variables for the values of the derivatives of all relevant orders at all relevant outcome levels, which allows for preference restrictions beyond the standard sign restrictions. Our systems of inequalities can be interpreted in terms of piecewise polynomial utility functions with a number of pieces that increases with the number of outcomes and the degree of SD. An empirical study analyzes the relevance of higher-order risk preferences for comparing a passive stock market index with actively managed stock portfolios in standard data sets from the empirical asset pricing literature.

1. Introduction

Stochastic Dominance (SD) ranks risky prospects based on general regularity conditions for decision making under risk (Hadar & Russell, 1969; Hanoch & Levy, 1969; Quirk & Saposnik, 1962; Rothschild & Stiglitz, 1970; Whitmore, 1970). Recent applications in OR/MS include Lizyayev and Ruszczynski (2012), Meskarian, Xu, and Fliege (2012), Roman, Mitra, and Zverovich (2013), Post and Kopa (2013), Dupacov and Kopa (2014), Hu, Mello, and Mehrotra (2014), Podinovski (2014), Armbruster and Delage (2015), Eeckhoudt, Fiori, and Gianin (2016), Longarela (2016), Meyer (1977), Post and Poti (2016) and Post and Kopa (2016).

The classical applications of SD compare a given prospect with a single alternative. More challenging applications involve multiple alternatives. In these cases, the concepts of 'SD optimality' (Bawa, 1975; Fishburn, 1974) and 'SD efficiency' (Kopa and Post, 2009; Kuosmanen, 2004; Post, 2003; Post and Versijp, 2007; Scaillet & Topaloglou, 2010; Lizyayev, 2012a; 2012b; Post, 2016; Longarela, 2016) apply. In these multivariate applications, a closed-form solution generally does not exist and numerical optimization is required.

Most studies focus on the first three degrees of SD ($N = 1, 2, 3$): first-degree SD (FSD), second-degree SD (SSD) and third-degree SD (TSD). In an ambitious attempt to generalize existing results, Post and Kopa (2013) develop systems of linear inequalities for general Nth degree SD (NSD; $N \geq 1$). With this general formulation, a large class of SD relations can be analyzed using Linear Programming (LP). The relevant LP problems are relatively small and convenient for large-scale applications, simulations and statistical resampling methods.

Despite its merits, the Post and Kopa (2013) approach is not exact but an approximation for SD optimality tests of degree $N \geq 3$ and SD efficiency tests of degree $N \geq 4$. Our study proposes a general revision of Post and Kopa (2013), aiming at stronger operational conditions for higher-degree SD relations. The revision applies to a range of SD relations; we revise even the simple case of pairwise TSD, which arises as a special case of SD optimality with two prospects and $N = 3$.

Our strongest results are obtained for SD optimality of degree $N = 1, 2, 3, 4$ and SD efficiency of degree $N = 2, 3, 4, 5$. For these SD relations, we find finite and exact systems of convex inequalities that can be implemented using LP or Convex Quadratic Programming (CQP). By comparison, the linear systems of Post and Kopa (2013) are exact only for optimality of degree $N = 1, 2$ and efficiency of degree $N = 2, 3$.

For optimality of degree $N \geq 5$ and efficiency of degree $N \geq 6$, our conditions are necessary but not sufficient. We do not consider this an important limitation. The arguments for restricting higher-order derivatives are less compelling than for lower-order derivatives. In addition, these restrictions generally have minimal effects on the flexibility to model the relevant utility levels (for optimality tests) or marginal utility levels (for efficiency tests).

* Corresponding author.
E-mail addresses: danielfang@163.com (Y. Fang), thierrypost@hotmail.com (T. Post).

http://dx.doi.org/10.1016/j.ejor.2017.03.035

Our analysis introduces model variables for the values of all $(N-1)$ relevant derivatives at all T relevant outcome levels. The additional model variables are not only needed for higher-degree SD relations but can also be used to impose restrictions on the values of the derivatives in addition to the standard restrictions on the signs. This feature is relevant for tests based on Decreasing Absolute Risk Aversion (DARA) SD (Vickson, 1975), Stochastic Dominance With respect to a Function (SDWRF; Meyer, 1977), Almost Stochastic Dominance (ASD; Leshno & Levy, 2002; Tzeng, Huang, & Shih, 2013) and Standard Stochastic Dominance' (StSD; Post, 2016).

One way to interpret our revision is that we use piecewise polynomial functions with a number of pieces that increases with the number of outcomes (T) and the relevant degree of SD (N). This characterization generalizes results by Hadar and Seo (1988) and Rothschild and Stiglitz (1970) on representative utility functions for pairwise comparison based on lower-degree SD rules. Similarly, Caballe and Pomansky (Section 4 1996), derive representative functions of infinite-degree SD, Kopa and Post (2009) and Post (2003) deal with the representation of FSD and SSD efficiency and Post, Fang, and Kopa (Section 3, 2015), with DARA SD optimality and efficiency.

We focus on SD optimality and efficiency tests for a given prospect. The problem of choosing a portfolio which stochastically dominates a given benchmark portfolio (Shalit & Yitzhaki, 1994; Dentcheva & Ruszczynski, 2003; Kuosmanen, 2004; Roman, Darby-Dowman, & Mitra, 2006; Scaillet & Topaloglou, 2010) is beyond the scope of this study. However, there exists a close link between these two topics. Notably, Kopa and Post (2015), Armbruster and Delage (2015) and Longarela (2016) construct SSD efficient portfolios by searching simultaneously over portfolio weights and utility functions using LP. Our results could be used to extend their results to TSD, fourth-degree SD (FOSD) and fifth-order SD (FISD) using CQP.

In an empirical study, we apply a range of portfolio efficiency tests to compare a passive stock market index with actively managed stock portfolios, in standard data sets from the empirical asset pricing literature. Our results show that the estimated pricing errors based on higher-order SD, as well as modifications of SSD based on SDWRF and ASSD, tend to be larger and more significant than standard mean-variance (MV) estimates, as a result of using pricing kernels that exclude arbitrage opportunities and account for systematic skewness. These findings add to the mounting evidence against market portfolio efficiency.

Appendix A presents formal proofs for our lemmas and propositions; Appendix B specifies the LP and CQP problems that we use for our numerical example in Section 7 and empirical application in Section 9.

2. Preliminaries

We use the general framework of Post and Kopa (2013). Their analysis considers $M \geq 2$ prospects with risky outcomes $x_1, \ldots, x_M \in \mathcal{D} := [A, B]$, $-\infty < A < B < +\infty$. The outcomes are treated as random variables with a discrete joint probability distribution characterized by R mutually exclusive and collectively exhaustive scenarios with probabilities $p_r > 0$, $r = 1, \ldots, R$.

We use $x_{i,\,r}$ for the outcome of prospect i in scenario r. We collect all possible outcomes in the joint support $Y := \{y : y = x_{i,r}\ i = 1, \ldots, M; r = 1, \ldots, R\}$, rank these values in ascending order, $y_1 \leq \ldots \leq y_S$, and use $p^*_{i;s} := \mathbb{P}[x_i = y_s] = \sum_{r=1}^{R} p_r \mathbb{I}(x_{i,r} = y_s)$, $i = 1, \ldots, M$; $s = 1, \ldots, S$.

Decision makers' preferences are described by von Neumann–Morgenstern utility functions. To implement SD of degree $N \geq 1$, we consider the following set of monotonic utility functions:

$$\mathcal{U}_N := \{u \in \mathcal{C}^N : (-1)^{n+1} u^n(x) \geq 0,\ n = 0, \ldots, N\}, \quad (1)$$

where $u^0(x) = u(x)$ and $u^n(x) := \partial^n u / \partial x^n$, $n = 1, \ldots, N$.

The economic interpretation of the restrictions on the first two derivatives is well-established: $u^1(x) \geq 0$ amounts to non-satiation and $u^2(x) \leq 0$ means risk aversion. The higher-order derivatives govern the higher-order risk preferences. Notably, $u^3(x) \geq 0$ means 'prudence', or skewness preference, and $u^4(x) \leq 0$ equals 'temperance', or kurtosis aversion. For discussions of the behavioral characterization and consequences of higher-order risk preferences, we refer to Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2016) and references therein.

The utility set $\mathcal{U}_N$ has two redundant but convenient features. First, the restriction $u(x) \leq 0$ is redundant, because utility analysis is location invariant. This restriction is however convenient because it implies $-u^1(x) \in \mathcal{U}_{N-1}$, which is a useful property in Section 6. Since the below definitions do not require the values of $u^N(x)$, the requirement that the Nth derivative is continuous is also redundant and $\mathcal{U}_N$ is equivalent to

$$\mathcal{U}^*_N := \{u \in \mathcal{C}^{N-1} : (-1)^n (u^n(y) - u^n(x)) \geq 0,\ n = 0, \ldots, N-1; y \geq x\}.$$

The use of $\mathcal{U}_N$ is however convenient to derive Lemma 1 without using sub-differential calculus. However, in Lemma 2 and Section 7, we use $\mathcal{U}^*_N$ to allow for jumps in the Nth derivative.

Definition 1 (Stochastic Dominance)**.** An evaluated prospect x_i, $i = 1, \ldots, M$, is dominated by alternative x_j, $j = 1, \ldots, M$, in terms of NSD, $N \geq 1$, if the former is strictly preferred to the latter for all permissible utility functions $u \in \mathcal{U}_N$:

$$\sum_{r=1}^{R} p_r u(x_{i,r}) < \sum_{r=1}^{R} p_r u(x_{j,r}) \Leftrightarrow \sum_{s=1}^{S} u(y_s)(p^*_{i,s} - p^*_{j,s}) < 0. \quad (2)$$

Various applications of SD consider a discrete choice set, $\mathcal{X}_0 := \{x_1, \ldots, x_M\}$, $M \geq 2$. This specification is relevant in welfare economics, where SD is widely applied following (Atkinson, 1970), because it is not possible to mix welfare distributions from different countries or periods. Similarly, in health economics, medical treatments are often indivisible and mutually exclusive.

Definition 2 (SD admissibility)**.** An evaluated prospect x_i, $i = 1, \ldots, M$, is admissible in terms of NSD, $N \geq 1$, if it is not dominated by any alternative combination $x \in \mathcal{X}_0$, in terms of NSD.

Algorithms for implementing this concept in an efficient manner were developed in Porter, Wart, and Ferguson (1973) and Bawa, Lindenberg, and Rafsky (1979). The admissibility concept however became obsolete after Bawa (1985) developed LP programs to implement a more powerful concept by Fishburn (1974):

Definition 2' (SD optimality)**.** An evaluated prospect x_i, $i = 1, \ldots, M$, is optimal in terms of NSD, $N \geq 1$, if it is preferred to every alternative $x \in \mathcal{X}_0$ for some permissible utility function $u \in \mathcal{U}_N$:

$$\sum_{r=1}^{R} p_r u(x_{i,r}) \geq \sum_{r=1}^{R} p_r u(x_r)\ \forall x \in \mathcal{X}_0 \Leftrightarrow \sum_{s=1}^{S} u(y_s)\left(p^*_{i,s} - p^*_{j,s}\right) \geq 0,\ j = 1, \ldots, M. \quad (3)$$

For $M = 2$, the two definitions are equivalent. However, for $M > 2$, Definition 2 is a necessary but not sufficient condition for Definition 2'. Put differently, a prospect can be non-optimal for all permissible utility functions without being dominated by any individual alternative.