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Decision Support

Robust and Pareto optimality of insurance contracts

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ABSTRACT

The optimal insurance problem represents a fast growing topic that explains the most efficient contract that an insurance player may get. The classical problem investigates the ideal contract under the assumption that the underlying risk distribution is known, i.e. by ignoring the parameter and model risks. Taking these sources of risk into account, the decision-maker aims to identify a robust optimal contract that is not sensitive to the chosen risk distribution. We focus on *Value-at-Risk* (VaR) and *Conditional Value-at-Risk* (CVaR)-based decisions, but further extensions for other risk measures are easily possible. The *Worst-case scenario* and *Worst-case regret* robust models are discussed in this paper, which have been already used in robust optimisation literature related to the investment portfolio problem. Closed-form solutions are obtained for the VaR Worst-case scenario case, while *Linear Programming (LP)* formulations are provided for all other cases. A caveat of robust optimisation is that the optimal solution may not be unique, and therefore, it may not be economically acceptable, i.e. Pareto optimal. This issue is numerically addressed and simple numerical methods are found for constructing insurance contracts that are Pareto and robust optimal. Our numerical illustrations show weak evidence in favour of our robust solutions for VaR-decisions, while our robust methods are clearly preferred for CVaR-based decisions.

1. Introduction

Finding the optimal insurance contract has represented a topic of interest in the actuarial science and insurance literature for more than 50 years. The seminal papers of Borch (1960) and Arrow (1963) had opened this field of research and since then, many papers discussed this problem under various assumptions on the risk preferences of the insurance players involved in the contract and how the cost of insurance (known as *premium*) is quantified. Specifically, the optimal contracts in the context of Expected Utility Theory are investigated amongst others in Kaluszka (2005), Kaluszka and Okolewski (2008) and Cai and Wei (2012). Extensive research has been made when the preferences are made via coherent risk measures (as defined in Artzner, Delbaen, Eber, and Heath, 1999; recall that CVaR is an element of this class) and VaR; for example, see Cai and Tan (2007), Balbás, Balbás, and Heras (2009); 2011), Asimit, Badescu, and Verdonck (2013b), Cheung, Sung, Yam, and Yung (2014) and Cai and Weng (2016) among others.

The choice of a risk measure is usually subjective, but VaR and CVaR represent the most known risk measures used in the insurance industry. Solvency II and Swiss Solvency Test are the regulatory regimes for all (re)insurance companies that operate within the European Union and Switzerland, respectively, and their capital requirements are solely based on VaR and CVaR. For these reasons and not only, these standard risk measures have received special attention by academics, practitioners and regulators, and therefore, vivid discussions have risen among them. VaR is criticised for its lack of sub-additivity and it may create regulatory arbitrage in an insurance group (see Asimit, Badescu, & Tsanakas, 2013a). A detailed discussion on possible regulatory arbitrages in a CVaR-based regime is provided in Koch-Medina and Munari (2016). A desirable property for a risk measure is the *elicitability*, which allows one to compare competitive forecasting methods, a property that VaR does have (see Gneiting, 2011). The lack of elicitability for CVaR has been adjusted via the *joint elicitability*, concept formalised in Fissler and Ziegel (2016), but earlier flagged out by Acerbi and Székely (2014). Robustness properties of a risk measure are also of great interest since they imply that the estimate is insensitive to

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data contamination. *Parameter risk* (uncertainty with parameter estimation) and *model risk* (uncertainty with model selection) are the two main sources of uncertainty in modelling. The robust statistic has its roots in the papers of Huber (1964) and Hampel (1968), which has been shown to be less appropriate in the context of risk management (see for example, Cont, Deguest, & Scandolo, 2010). A more informative discussion is given in the next section due to its length. Finally, a summary of all properties exhibited by the two risk measures is detailed in the comprehensive work of Emmer, Kratz, and Tasche (2015), but the general conclusion is that there is no evidence for global advantage of one risk measure against the other.

Whenever the model and parameter risks are present, it is prudent to consider insurance contracts that are optimal under a set of plausible models and this is precisely what *robust optimisation* does. It is a vast area of research with applications in various fields and a standard reference is Ben-Tal, El Ghaoui, and Nemirovski (2009), while comprehensive surveys can be found in Ben-Tal and Nemirovski (2008), Bertsimas, Brown, and Caramanis (2011) and Gabrel, Murat, and Thiele (2014).

The aim of the paper is to identify the optimal insurance contract under the model/parameter risk in the robust optimisation sense and understand how robust these solutions are from the practical point of view. That is, we aim to explain how large the uncertainty set should be for relatively small or medium sized historical data sets as is expected in insurance practice. At the same time, since the insurance contract is in fact a risk allocation, it is of great interest to find whether or not our robust contracts are Pareto optimal. Robust optimisation may lead to inefficient risk allocations, i.e. not Pareto optimal, which are clearly not acceptable, and special attention is given to this issue by providing a simple methodology to overcome such caveats of robust optimisation. Our numerical illustrations have shown weak evidence in favour of our robust solutions for VaR-based decisions, which is not surprising due to the erratic behaviour of VaR. On the contrary, CVaR-based decisions are more robust via robust optimisation than using statistical methods, which can be explained by the fact that CVaR takes into account some part of the tail risk as opposed to VaR. Either Worst-case scenario or regret robust optimisations is preferred (comparing to the classical statistical methods) for less (statistically) robust risk measures that are purely tail risk measures, where the estimation is based on a small portion of the sample that explains only the tail risk. We also find that the Worst-case optimisation is once again advantageous even for risk measures that are sensitive to the entire sample, i.e. are not only based on the tail risk.

The structure of the paper is as follows: the next section contains the necessary background and the mathematical formulation of our problems, while Sections 3 and 4 investigate the VaR and CVaR-based optimal insurance contracts, but also discuss simple extensions for distortion risk measures when the moral hazard is removed; these robust solutions are further investigated in Section 5 to becoming Pareto optimal as well; extensive numerical examples are elaborated in Section 6, which help in justifying our conclusions summarised in Section 7.

2. Background and problem definition

2.1. Optimal insurance

An insurance contract represents a risk transfer between two parties, *insurance buyer* (or simply *buyer*) and *insurance seller* (or simply *seller*). When the buyer is also an insurance company, then the transfer becomes a *reinsurance contract* and the seller is called *reinsurer*. Let $X \geq 0$ be the total amount that the buyer is liable to pay in the absence of any risk transfer. In addition, the seller agrees to pay $R[X]$, the amount by which the entire loss exceeds the buyer's amount, $I[X]$, and clearly we have $I[X]+R[X]=X$. The most common risk transfers are the Proportional and Stop-loss contracts for which $I[X]=cX$ (with $0 \leq c \leq 1$) and $I[X]=\min\{X, M\}$, respectively. Note that in order to avoid moral hazard issues (both players are incentivised to reduce the overall risk, i.e. I and R are non-decreasing functions), $I, R \in \mathcal{C}^{co}$, where

$$\mathcal{C}^{co} = \{f \text{ is non-decreasing} \mid 0 \leq f(x) \leq x, |f(x)-f(y)| \leq |x-y| \text{ for all } x, y \in \Re\}.$$

The comonotonic risk transfers (as defined above) are omnipresent in practice, but it is not always the case and the mathematical formulation of the feasibility set becomes

$$\mathcal{C} = \Big\{f \mid 0 \leq f(x) \leq x \text{ for all } x \in \Re\Big\}.$$

Let P be the insurance premium, and it is further assumed that any feasible contract satisfies $0 \leq P \leq \overline{P}$, where $\overline{P}$ represents a maximal amount of premium that the buyer would accept to pay. If the loss distribution is known, then the premium calculations are possible via certain rules, known as *premium principles*. A concise review of premium principles can be found in Young (2004). Specifically, if $\mathcal{P}$ is the probability measure for X, then $P \geq \omega_0 + (1+\theta)\mathbb{H}_{\mathcal{P}}\big(R[X]\big)$, where $\omega_0 \geq 0$ represents some fixed/administrative costs, $\theta \geq 0$ is the *risk loading* parameter/factor, and $\mathbb{H}$ is a monotone functional on the space of non-negative random variables that depends on the seller's choice of premium principle. The monotonicity property is of practical importance and it means that if two random losses satisfy $Y \leq Z$, then $\mathbb{H}_{\mathcal{P}}(Y) \leq \mathbb{H}_{\mathcal{P}}(Z)$. A commonly encountered premium principle is the *distortion premium principle* (see Wang, Young, & Panjer, 1997),

$$\mathbb{H}_{\mathcal{P}}(Y) = \int_0^\infty g\big(\mathcal{P}(Y>y)\big)\,dy \tag{2.1}$$

for any non-negative loss random variable Y, where g: $[0, 1] \to [0, 1]$ is non-decreasing with $g(0)=0$ and $g(1)=1$ known as *distortion function*. When the distortion function is taken to be the identity function, we obtain the *expected value premium principle*, which is standard in the insurance industry. The mathematical formulation of the optimal insurance problem becomes

$$\min_{(R,P)\in\mathcal{C}\times\Re} \{\rho_{\mathcal{P}}(X-R[X]+P)\}, \text{ s.t. } \omega_0+(1+\theta)\mathbb{H}_{\mathcal{P}}(R[X]) \leq P \leq \overline{P}, \tag{2.2}$$

where $\rho_{\mathcal{P}}$ is a risk measure chosen by the buyer to order its preferences to risk. As explained in Section 1, it is first assumed in this paper that $\rho_{\mathcal{P}} \in \{\text{VaR}, \text{CVaR}\}$. Recall that the lower script $\mathcal{P}$ indicates the probability measure under which the risk measurement is made. The VaR of a loss variable Y at a confidence level $\alpha \in (0, 1)$, is given by $\text{VaR}_\alpha(Y;\mathcal{P}) = \inf_{y\in\Re}\{\mathcal{P}(Y \leq y) \geq \alpha\}$. Note that VaR_α is representable as in (2.1) with $g(t) = \mathbb{I}_{\{t>1-p\}}$, where $\mathbb{I}_A$ represents the indicator operator that assigns the value 1 if A is true and 0 otherwise. The CVaR risk measure is defined in Rockafeller and Uryasev (2000) as follows:

$$\text{CVaR}_\alpha(Y;\mathcal{P}) = \inf_{t\in\Re}\left\{t+\frac{1}{1-\alpha}\mathbf{E}_{\mathcal{P}}\big(Y-t\big)_+\right\}, \text{ where } (t)_+ = \max(t, 0). \tag{2.3}$$

Alternative representations are known in the literature (see for example, Acerbi & Tasche, 2002) and one of them is as in (2.1) with $g(t) = \frac{t}{1-\alpha} \wedge 1$.

Due to the monotonicity property of VaR, CVaR and the functional $\mathbb{H}$, (2.2) becomes much simpler when removing the economic constraint $P \leq \overline{P}$ and it has been investigated under various sets of assumptions. Recently, Cheung and Lo (2017) included the latter constraint and analytically solved (2.2) for a large class