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Short Communication

Non-cooperative two-stage network DEA model: Linear vs. parametric linear

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ABSTRACT

In the data envelopment analysis (DEA) literature, linear fractional non-cooperative network DEA models for two-stage network structures are often transformed into parametric linear models. The transformed parametric linear models are then solved by computing a series of linear models when the parameter is varied. For example, Wu, Zhu, Ji, Chu and Liang (2016) provide a linear fractional non-cooperative DEA model for analyzing the reuse of undesirable intermediate outputs in a two-stage production process with a shared resources and feedback. They transformed the linear fractional model into a parametric linear model. Such approaches do not guarantee that the global optimal solution is found. We show that (variants of) linear fractional non-cooperative network DEA models can be directly transformed into a linear programing model, without the need for solving parametric linear models. This greatly reduces the computational burden and the global optimal solution is always guaranteed.

1. Introduction

In recent years, multi-stage or network structures have been an important area in data envelopment analysis (DEA) (Cook & Zhu, 2014). One of the techniques in DEA network modeling is based upon game theory concept. For example, Liang, Yang, Cook, and Zhu (2006) propose additive efficiency models and non-cooperative efficiency models. Liang, Cook, and Zhu (2008) formally introduce the technique for modeling two-stage network decision making units (DMUs) from the perspective of the non-cooperative or leader–follower and cooperative games. In a similar manner, Wu et al. (2016) provide non-cooperative and cooperative models for analyzing the reuse of undesirable intermediate outputs in a two-stage production process with a shared resource and feedback.

Usually, the non-cooperative network DEA models are linear fractional and are solved by transforming the DEA-type linear fractional programs into parametric linear models which are then solved using heuristic method. We, however, show that those linear fractional non-cooperative network DEA models can be directly transformed into a linear program by using only one Charnes–Cooper transformation (Charnes & Cooper, 1962).

In the next section, we use the non-cooperative model of Wu et al. (2016) as example to show that the linear fractional model can be converted into a linear model using one Charnes–Cooper transformation. We further show that variants of linear fractional non-cooperative network DEA network models for the general two-stage network structures can be transformed into linear models. Conclusions are given in the last section.

2. Non-cooperative model with a shared resource and feedback

Fig. 1 presents the two-stage network process studied in Wu et al. (2016). Using the notations from Wu et al. (2016), we assume that there are a set of n DMUs and that for each DUM$_j$ ($j=1,2,\ldots,n$), the stage 1 consumes m inputs $X_{ij}(i=1,2,\ldots,m)$, G inputs $H_{gj}(g=1,2,\ldots,G)$ and K inputs $Z_{kj}=Z_{kj}^1+Z_{kj}^2(k=1,2,\ldots,K)$ to produce s desirable outputs $Y_{rj}(r=1,2,\ldots,s)$ and D undesirable outputs $F_{dj}(d=1,2,\ldots,D)$. In addition, the desirable outputs Y_{rj} leave the system but the undesirable outputs F_{dj} may be disposed in the stage 2 by using P inputs $R_{pj}(p=1,2,\ldots,P)$ and m inputs m inputs $X_{ij}(i=1,2,\ldots,m)$ to obtain K desirable outputs $Z_{kj}^2(k=1,2,\ldots,K)$, which serve as input resources of the stage 1. Shared inputs $x_{ij}$are divided into $\alpha_{ij}x_{ij}$ and $(1-\alpha_{ij})x_{ij}$, where $0\le\alpha_{ij}\le 1$, which correspond to the portions of shared inputs used by stages 1 and 2, respectively. As in Cook and Hababou (2001) and Chen, Du, Sherman, and Zhu (2010), Wu et al. (2016) assume that α_{ij} has upper and lower bounds as $L_i\le\alpha_{ij}\le U_i$.

$$\mathrm{DMU}_j,\ \ j=1,\ 2,\ldots,n$$

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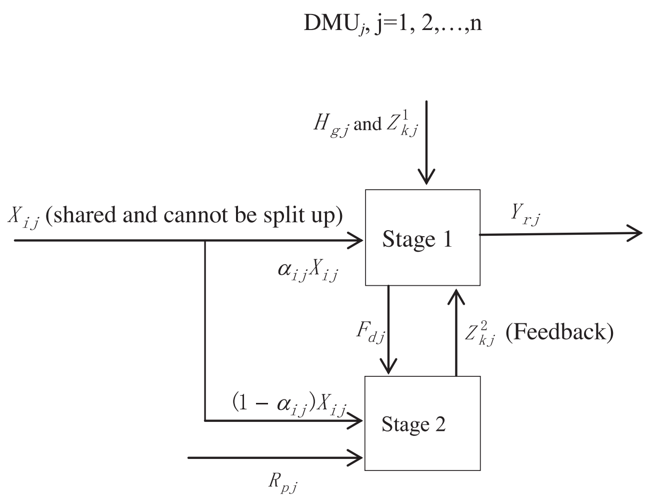

Fig. 1. Two-stage network structure in Wu et al. (2016).

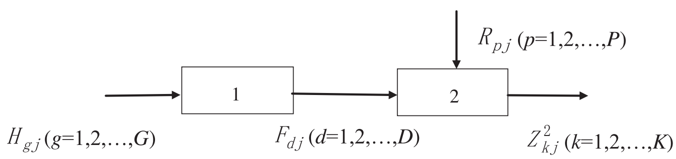

Fig. 2. Two-stage network structure in Liang et al. (2006).

Let us consider the non-cooperative model of Wu et al. (2016) where the first stage is assumed to be the leader. The efficiency of the first stage (E^{1*}_{1o}) for a specific DMU$_o$ is calculated first using a CCR-type linear model (Charnes, Cooper, & Rhodes, 1978). The following linear fractional model is established to obtain the follower stage's efficiency when the first stage's efficiency is fixed at E^{1*}_{1o} as a constraint.

$$
\begin{aligned}
\max\quad & E^1_{2o} = \frac{\sum_{k=1}^{K}\pi_k Z^2_{ko}}{\sum_{i=1}^{m} v_i((1-\alpha_{io})X_{io} + \sum_{p=1}^{P}\eta_p R_{po} + \sum_{d=1}^{D}\varphi_d F_{do}} \\
\text{s.t.}\quad & \frac{\sum_{r=1}^{s} u_r Y_{ro} - \sum_{d=1}^{D}\varphi_d F_{do}}{\sum_{i=1}^{m} v_i\alpha_{io}X_{io} + \sum_{k=1}^{K}\pi_k Z^1_{ko} + \sum_{k=1}^{K}\pi_k Z^2_{ko} + \sum_{g=1}^{G} w_g H_{go}} = E^{1*}_{1o} \\
& \frac{\sum_{r=1}^{s} u_r Y_{rj} - \sum_{d=1}^{D}\varphi_d F_{dj}}{\sum_{i=1}^{m} v_i\alpha_{ij}X_{ij} + \sum_{k=1}^{K}\pi_k Z^1_{kj} + \sum_{k=1}^{K}\pi_k Z^2_{kj} + \sum_{g=1}^{G} w_g H_{gj}} \le 1\ \forall j \\
& \frac{\sum_{k=1}^{K}\pi_k Z^2_{kj}}{\sum_{i=1}^{m} v_i((1-\alpha_{ij})X_{ij} + \sum_{p=1}^{P}\eta_p R_{pj} + \sum_{d=1}^{D}\varphi_d F_{dj}} \le 1\ \forall j \\
& L_i \le \alpha_{ij} \le \mathrm{U}_i,\quad \forall \mathrm{i}, \mathrm{j} \\
& u_r, v_i, \varphi_d, \pi_k, w_g, \eta_p \ge 0,\quad \forall r, i, d, k, g, p
\end{aligned}
\tag{1}
$$

Note that there is a typo in Wu et al. (2016) models ((5) and (7)) where $\alpha_{ij}x_{ij}$ of shared inputs are used by stage 2 and $(1-\alpha_{ij})x_{ij}$ of shared inputs are used by stage 1. In fact, $\alpha_{ij}x_{ij}$ should be associated with stage 1 and $(1-\alpha_{ij})x_{ij}$ should be associated with stage 2. We have corrected that in the above model (1).

To solve the model (1), Wu et al. (2016) apply simultaneously two Charnes–Cooper transformations and obtain a parametric linear model. Then the parametric model is solved in a series of linear models by varying the parameter. Such an approach cannot guarantee that the global solution is always obtained.

In fact, model (1) can be directly transformed into a linear model by only one Charnes–Cooper transformation.

Specifically, let $t = \frac{1}{\sum_{i=1}^{m} v_i((1-\alpha_{io})X_{io} + \sum_{p=1}^{P}\eta_p R_{po} + \sum_{d=1}^{D}\varphi_d F_{do}}$ and set $\tilde{u}_r = tu_r, \tilde{v}_i = tv_i,\ \tilde{\varphi}_d = t\varphi_d,\ \tilde{\pi}_k = t\pi_k, \tilde{w}_g = tw_g, \tilde{\eta}_p = t\eta_p$, then model (1) is converted into the following linear model

$$
\begin{aligned}
\max\quad & E^1_{2o} = \sum_{k=1}^{K}\tilde{\pi}_k Z^2_{ko} \\
\text{s.t.}\quad & \sum_{r=1}^{s}\tilde{u}_r Y_{ro} - \sum_{d=1}^{D}\tilde{\varphi}_d F_{do} \\
& \quad - E^{1*}_{1o}\left[\sum_{i=1}^{m}\xi_{io}X_{io} + \sum_{k=1}^{K}\tilde{\pi}_k Z^1_{ko} + \sum_{k=1}^{K}\tilde{\pi}_k Z^2_{ko} + \sum_{g=1}^{G}\tilde{w}_g H_{go}\right] = 0 \\
& \sum_{r=1}^{s}\tilde{u}_r Y_{rj} - \sum_{d=1}^{D}\tilde{\varphi}_d F_{dj} - \sum_{i=1}^{m}\xi_{ij}X_{ij} - \sum_{k=1}^{K}\tilde{\pi}_k Z^1_{kj} \\
& \quad - \sum_{k=1}^{K}\tilde{\pi}_k Z^2_{kj} - \sum_{g=1}^{G}\tilde{w}_g H_{gj} \le 0\forall \mathrm{j} \\
& \sum_{k=1}^{K}\tilde{\pi}_k Z^2_{kj} - \sum_{i=1}^{m}(\tilde{v}_i - \xi_{ij})X_{ij} - \sum_{p=1}^{P}\tilde{\eta}_p R_{pj} - \sum_{d=1}^{D}\tilde{\varphi}_d F_{dj} \le 0\forall \mathrm{j} \\
& \sum_{i=1}^{m}(\tilde{v}_i - \xi_{io})X_{io} + \sum_{p=1}^{P}\tilde{\eta}_p R_{po} + \sum_{d=1}^{D}\tilde{\varphi}_d F_{do} = 1 \\
& \tilde{v}_i L_i \le \xi_{ij} \le \tilde{v}_i \mathrm{U}_i,\quad \forall \mathrm{i}, \mathrm{j} \\
& \tilde{u}_r, \tilde{v}_i, \tilde{\varphi}_d, \tilde{\pi}_k, \tilde{w}_g, \tilde{\eta}_p \ge 0,\quad \forall r, i, d, k, g, p
\end{aligned}
\tag{2}
$$

where $\xi_{ij} = \alpha_{ij}\tilde{v}_i$. In a similar manner, we can convert the linear fractional non-cooperative model into a linear model when stage 2 is assumed to be the leader.

3. General non-cooperative model

We have just demonstrated that linear fractional non-cooperative models in Wu et al. (2016) can be converted into linear programing models. In fact, all linear fractional non-cooperative or leader–follower two-stage network DEA models can be converted into linear programing models without the need for solving parametric models. Suppose the efficiency of the leader is denoted by $E^{leader*}_o$ and the efficiency of the follower is denoted by $E^{follower*}_o$. The efficiency ($E^{leader*}_o$) of the leader stage can be obtained by the a linear CCR-type model. Then the efficiency ($E^{follower*}_o$) of the follower stage is calculated by setting the efficiency of leader stage equal to $E^{leader*}_o$ as a constraint.

$$
\begin{aligned}
& E^{follower*}_o = \max\ E^{follower}_o \\
s.t.\quad & E^{\text{leader}}_o = E^{leader*}_o \\
& E^{\text{leader}}_j \le 1\ \forall j \\
& E^{\text{follower}}_j \le 1\ \forall j
\end{aligned}
\tag{3}
$$

Note that E^{leader}_j and $E^{follower}_j$ are DEA efficiency ratios that have weighted inputs and weighted outputs linear terms in both the numerators and denominators. Consequently, each linear fractional constraint of model (3) can be easily converted into a linear constraint. Moreover, the objective function can be transformed into a linear form by the Charnes–Cooper transformation. Meanwhile, every decision variable of the model (3) can be transformed via a positive scalar. Then the non-cooperative model (3) can be directly transformed into a linear program.

For example, Fig. 1 presents a specific type of two-stage network structure studied in Liang et al. (2006). In fact, the two-stage network process in Fig. 1 can be obtained via removing the shared inputs x_{ij}, desirable outputs Y_{rj} and discarding the undesirable attributes of F_{dj} and the feedback attributes of Z^2_{kj} from the process in Wu et al. (2016). Therefore, it is evident that the non-cooperative model of Liang et al. (2006) can also be solved by just using one linear model rather than a parametric linear program.

4. Concluding remarks

We have showed that linear fractional non-cooperative models for two-stage network DEA structures can be directly transformed