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Shape optimization using the cut finite element method[☆]

Erik Burman[a], Daniel Elfverson[b,*], Peter Hansbo[c], Mats G. Larson[a], Karl Larsson[a]

[a] *Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK*
[b] *Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden*
[c] *Department of Mechanical Engineering, Jönköping University, SE-551 11 Jönköping, Sweden*

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Highlights

- Shape optimization using a cut finite element method for linear elasticity.
- The evolving geometry is described using a level-set representation.
- For higher order finite elements we use a refined grid for the level-set.
- The geometry is evolved by solving a transport problem based on the shape derivative.

Abstract

We present a cut finite element method for shape optimization in the case of linear elasticity. The elastic domain is defined by a level-set function, and the evolution of the domain is obtained by moving the level-set along a velocity field using a transport equation. The velocity field is the largest decreasing direction of the shape derivative that satisfies a certain regularity requirement and the computation of the shape derivative is based on a volume formulation. Using the cut finite element method no re-meshing is required when updating the domain and we may also use higher order finite element approximations. To obtain a stable method, stabilization terms are added in the vicinity of the cut elements at the boundary, which provides control of the variation of the solution in the vicinity of the boundary. We implement and illustrate the performance of the method in the two-dimensional case, considering both triangular and quadrilateral meshes as well as finite element spaces of different order.

Keywords: CutFEM; Shape optimization; Level-set; Fictitious domain method; Linear elasticity

1. Introduction

Optimization of elastic structures is an important and active research field of significant interest in engineering. There are two common approaches to represent the domain which we seek to optimize: (i) *A density function.* This

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* Corresponding author.
E-mail addresses: e.burman@ucl.ac.uk (E. Burman), daniel.elfverson@umu.se (D. Elfverson), peter.hansbo@ju.se (P. Hansbo), mats.larson@umu.se (M.G. Larson), karl.larsson@umu.se (K. Larsson).

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approach, common in topology optimization [1,2], is very general and computationally convenient, but the boundary representation is not sharp and thus typically fine grids and low order approximation spaces are employed. (ii) *An implicit or explicit representation of the boundary.* This approach is common in shape optimization [3] where the boundary is typically described by a level-set function or a parametrization, but topological changes can also be handled for instance using an implicit level-set representation of the boundary. Given the boundary representation we need to generate a discretization of the domain when it is updated. This can be done using a standard meshing approach based on mesh motion and/or re-meshing or alternatively using a fictitious domain method, see [4–6] for different approaches.

In this contribution we focus on the fictitious domain approach using the recently developed cut finite element method CutFEM [7,8], extending our previous work on the Bernoulli free boundary value problem [9] to linear elasticity. The key components in CutFEM are: (i) Use of a fixed background mesh and a sharp boundary representation that is allowed to cut through the background mesh in arbitrary fashion. (ii) Weak enforcement of the boundary conditions. (iii) Stabilization of the cut elements in the vicinity of the boundary using a consistent stabilization term which leads to optimal order accuracy and conditioning of the resulting algebraic system of equations. CutFEM also allows higher order finite element spaces and rests on a solid theoretical foundation including stability bounds, optimal order a priori error bounds, and optimal order bounds for the condition numbers of the stiffness and mass matrices, see [8] and the references therein.

In order to support large changes in the shape and topology of the domain during the optimization process we employ a level-set representation of the boundary. The evolution of the domain is obtained by moving the level-set along a velocity field using a Hamilton–Jacobi transport equation, see [4,10]. The velocity field is the largest decreasing direction of the shape derivative that satisfies a certain regularity requirement together with a boundary conditions on the boundary of the design volume. The computation of the shape derivative is based on a volume formulation, see [11,12] for similar approaches. In this context CutFEM provides an accurate and stable approximation of the linear elasticity equations which completely avoids the use of standard meshing procedures when updating the domain. In this contribution we focus on standard Lagrange elements, but a wide range of elements may be used in CutFEM, including discontinuous elements, isogeometric elements with higher order regularity, and mixed elements.

When using a higher order finite element space in shape optimization we use a finer representation of the domain than the computational mesh, i.e., the level-set representing the geometry is defined on a finer mesh. For computational convenience and efficiency we use a piecewise linear geometry description on the finer grid. This approach of course leads to loss of optimal order but the main purpose of using the finer grid is to allow the domain to move more freely on the refined grid despite using larger higher order elements for the approximation of the solution field. If necessary, once a steady design has been found, a more accurate final computation can be done which uses a higher order geometry description. Our approach leads to a convenient and efficient implementation since the solution to the elasticity equations and the level-set function are represented using the fixed background mesh and a uniform refinement thereof.

An outline of the paper is as follows: in Section 2 we formulate the equations of linear elasticity and the optimization problem, in Section 3 we formulate the CutFEM, in Section 4 we recall the necessary results from shape calculus, in Section 5 we formulate the transport equation for the level-set and the optimization algorithm, and finally in Section 6 we present numerical results.

2. Model problem

2.1. Linear elasticity

Let $\Omega \in \mathbb{R}^d$, for $d = 2, 3$, be a bounded domain with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ such that $\Gamma_D \cap \Gamma_N = \emptyset$, and let n be the exterior unit normal to $\partial\Omega$. Assuming a linear elastic isotropic material the constitutive relationship between the symmetric stress tensor σ and the strain tensor ϵ is given by Hooke's law

$$\sigma = 2\mu\epsilon + \lambda\,\mathrm{tr}(\epsilon)I \tag{2.1}$$

where μ and λ are the Lamé parameters and I is the $d \times d$ identity matrix. Also, assuming small strains we may use the linear strain tensor $\epsilon(u) = (\nabla u + \nabla u^T)/2$, where u is the displacement field, as a strain measure. In this expression