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Reduced integration at superconvergent points in isogeometric analysis

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Highlights

- We propose a new reduced quadrature rule for IGA based on variational collocation.
- Integration is performed on the weighted residual form and not on the variational form.
- The quadrature points are superconvergent points.
- Two quadrature points per parametric direction are used for any (odd) degree.
- Results are very close to those obtained with Galerkin and accurate quadrature.

Abstract

We propose a new reduced integration rule for isogeometric analysis (IGA) based on the concept of variational collocation. It has been recently shown that, if a discrete space is constructed by smooth and pointwise non-negative basis functions, there exists a set of points – named Cauchy–Galerkin (CG) points – such that collocation performed at these points can reproduce the Galerkin solution of various boundary value problems exactly. Since CG points are not known a-priori, estimates are necessary in practice and can be found based on superconvergence theory. In this contribution, we explore the use of estimated CG points (i.e. superconvergent points) as numerical quadrature points to obtain an efficient and stable reduced quadrature rule in IGA. We use the weighted residual formulation as basis for our new quadrature rule, so that the proposed approach can be considered intermediate between the standard (accurately integrated) Galerkin variational formulation and the direct evaluation of the strong form in collocation approaches. The performance of the method is demonstrated by several examples. For odd degrees of discretization, we obtain spatial convergence rates and accuracy very close to those of accurately integrated standard Galerkin with a quadrature rule of two points per parametric direction independently of the degree.

Keywords: Variational collocation; Isogeometric analysis; Galerkin method; Collocation method; Superconvergent points

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1. Introduction

Isogeometric analysis (IGA) is a recently developed framework for the discretization of partial differential equations (PDEs). Here, in contrast to classical finite element methods (FEMs), smooth basis functions commonly employed in Computer Aided Design (CAD) are used, therefore providing an exact description of the geometry in the analysis model. IGA benefits from the higher continuity and regularity of the underlying basis functions and has shown significant advantages in various fields of application, especially a higher accuracy per degree-of-freedom, compared to conventional $\mathcal{C}^0$ FEMs. For a general overview about IGA, the reader is referred to [1].

The efficient numerical implementation is still an issue and one field of attention in current research in IGA. Standard Gauss quadrature rules are not well-suited for IGA since they do not exploit the higher continuity of the shape functions. This fact leads to a large computational effort for assembly of IGA matrices, especially when basis functions of high degree are used. Therefore reduced and optimal quadrature schemes already well established for FEMs, see Strang & Fix [2], have attracted revitalized attention in IGA, e.g. in [3–9]. The authors of [10] also performed parallelized IGA computations on GPUs to decrease the total computation time. Recently, Calabrò et al. [11] presented a novel algorithm for the formation of IGA matrices based on weighted quadrature, leading to remarkable time savings.

Besides these approaches, collocation methods have gained a great interest in the computational mechanics community, since they take full advantage of the higher regularity of the basis functions used in IGA. Collocation methods are based on the discretization of the strong form of the governing PDEs. They have been successfully applied to a broad range of problems relevant to the computational mechanics community, see e.g. [12–16]. In the majority of publications regarding isogeometric collocation methods, the Greville abscissae [17] are chosen as locations of the collocation points. In some cases also the Demko abscissae [18] are used, generally leading to comparable results as with the Greville abscissae. For both choices, numerical studies (e.g. in [19,20]) indicated for isogeometric collocation a spatial convergence rate in the L^2 norm of the error equal to the polynomial degree p for even degrees and to $p - 1$ for odd degrees.

A recent study [21] showed that, if a discrete space is constructed by smooth and pointwise non-negative basis functions, there exists a set of points – called Cauchy–Galerkin (CG) points – such that a collocation scheme performed at these points can reproduce the Galerkin solution of various boundary value problems exactly. The approach, which was given by its authors the name of variational collocation, is obviously suitable for the basis functions applied in IGA. The exact CG points were calculated numerically in [21], based on a given Galerkin solution. Since the latter is not known a priori, estimates need to be provided independent of the unknown Galerkin solution. These estimates can be obtained via superconvergence theory and therefore, in the following, the estimated CG points will be referred to as superconvergent points. The proposed estimate is suitable for uniform Bézier meshes and the investigation of an extension to the non-uniform case is in progress.

The number of superconvergent points is larger than the number of control points. In [22], the same points had been found and their complete set employed within a least-squares isogeometric collocation method. Conversely, in [21] only a subset of these points was used, thus avoiding the solution of an overdetermined system and leaving the structure of isogeometric collocation unaltered. Collocating at an alternating subset of points was shown to achieve convergence rates of p in the L^2 norm for both even and odd polynomial degrees. By using a subset with local symmetry [23], optimal convergence rates of $p + 1$ in the L^2 norm could be obtained for odd degrees. As of today, the mathematical reasons of this observed behavior are unknown.

In this paper, as an alternative to the collocation schemes mentioned before, we combine the collocation and Galerkin methods by applying the concept of variational collocation to a weighted residual approach. The superconvergent points are used as integration points for a new reduced quadrature rule. Instead of applying this rule to the Galerkin variational formulation of the considered numerical problems, we use directly the weighted residual formulation and apply to it the new quadrature rule. The chosen approach can thus be seen as intermediate between the Galerkin variational formulation and the direct evaluation of the strong form in collocation approaches.

This paper is organized as follows: In Section 2 the basic notions on isogeometric basis functions are briefly reported. In Section 3 we introduce the concept of variational collocation and the estimation of the CG points via superconvergence theory. In addition we point out how this concept can be applied to derive a reduced quadrature rule for weighted residual methods. Several numerical examples including a scalar problem, two-dimensional linear elasticity problems, and a generalized eigenvalue problem are investigated in Section 4. Conclusions are drawn and future research directions are outlined in Section 5.