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Symbolic computation of equivalence transformations and parameter reduction for nonlinear physical models

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ABSTRACT

An efficient systematic procedure is provided for symbolic computation of Lie groups of equivalence transformations and generalized equivalence transformations of systems of differential equations that contain arbitrary elements (arbitrary functions and/or arbitrary constant parameters), using the software package `GeM` for `Maple`. Application of equivalence transformations to the reduction of the number of arbitrary elements in a given system of equations is discussed, and several examples are considered. The first computational example of generalized equivalence transformations where the transformation of the dependent variable involves an arbitrary constitutive function is presented.

As a detailed physical example, a three-parameter family of nonlinear wave equations describing finite anti-plane shear displacements of an incompressible hyperelastic fiber-reinforced medium is considered. Equivalence transformations are computed and employed to radically simplify the model for an arbitrary fiber direction, invertibly reducing the model to a simple form that corresponds to a special fiber direction, and involves no arbitrary elements.

The presented computation algorithm is applicable to wide classes of systems of differential equations containing arbitrary elements.

1. Introduction

The majority of contemporary mathematical models, in particular, physical models written in terms of partial or ordinary differential equations (PDEs, ODEs), involve constant or variable parameters. These parameters, also called arbitrary or constitutive elements, appear in the equations in the form of arbitrary constants (*constitutive parameters*) that assume values in certain ranges, and/or arbitrary functions (*constitutive functions*) belonging to certain classes. Equivalent formulations of a model at hand, involving fewer constitutive parameters and/or reduced forms of constitutive functions, can lead to a significant simplification of the analysis of the model, in particular, when such analysis involves classifications, such as local symmetry or conservation law classifications. Equivalent formulations of a class of models may be systematically sought using *equivalence transformations*.

Equivalence transformations of a family of DE (differential equation) systems are transformations of problem variables and arbitrary elements that map every DE system from the given family into another system from the same family. As a basic illustration, consider a set of $(1+1)$-dimensional nonlinear diffusion equations for $u(x,t)$, given by

$$u_t = c^2(u)u_{xx}, \tag{1.1}$$

where $c(u)$ is an arbitrary constitutive function. (In (1.1) and throughout the paper where appropriate, subscripts denote derivatives.) The family (1.1) admits, for example, an obvious discrete equivalence transformation $c^*(u) = -c(u)$; in particular, any solution $u(x,t)$ of (1.1) is mapped a solution of a PDE $u_{tt} = (c^*(u))^2u_{xx}$. Scalings and translations are the simplest kinds of continuous equivalence transformations. In particular, transformations that re-scale variables using their typical values, and thus map a problem into a dimensionless one, or scaling transformations in general, have a long history, and are commonly used to simplify and non-dimensionalize DE problems. For example, the PDE family (1.1) clearly admits scaling and translation-type equivalence transformations

$$x^* = A_3x + A_1, \quad t^* = A_4t + A_2, \quad u^*(x^*,t^*) = A_5u(x,t), \quad c^*(u^*) = \frac{A_3^2}{A_4}c(u), \tag{1.2}$$

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$A_1, \ldots, A_5 \in \mathbb{R}, A_4 > 0$, so that a solution $u(x, t)$ of a PDE (1.1) with a constitutive function $c(u)$ is mapped into a solution $u^*(x^*, t^*)$ of a PDE

$$u^*_{t^*} = (c^*(u^*))^2 u^*_{x^*x^*},$$

a member of the family (1.1) with a different constitutive function, as long as $A_3^2 \neq A_4$.

The notion of equivalence transformations is closely related to that of *symmetries* of a system of differential equations. A symmetry is a transformation that leaves invariant the solution manifold of a given system, i.e., maps any solution of the system into a solution of the same system. Due to the general topological nature of this definition, a systematic way to calculate *all* symmetries of a given DE system, and similarly, *all* equivalence transformations of a given family of DE systems, is not available. However, equivalence transformations of particular kinds, such as Lie groups of local and nonlocal equivalence transformations, can indeed be computed systematically.

Work on equivalence transformations was initiated in [1]. Various examples, extensions, and applications of the notion of equivalence transformations appear in the literature, such as [2–9] and references therein. A broad review and work on equivalence transformations and some related topics, including both theoretical and computational aspects, is contained in [3]. A related problem, in a certain sense complementary to finding equivalence transformations of a given DE family, is known as the *Cartan equivalence problem* [10–12]; it consists in finding criteria for DEs within the family to be connected by a change of variables from a prescribed transformation group (see, e.g., [3] and references therein for a discussion and recent developments). Significant progress has recently been made in the application of the Cartan's method of equivalence (MoE) to differential equations (see, e.g., Refs. [13–19]). A symbolic implementation of routines for the automated MoE-related computations is contained in the standard `Maple` package `DifferentialGeometry` [20].

The current manuscript presents an algorithmic procedure to calculate Lie groups of local equivalence transformations of a given DE system using a symbolic symmetry and conservation law computation package `GeM` for `Maple` [21–23] developed by the author. A brief review of the notion of equivalence transformations, relations between Lie groups of point equivalence transformations and Lie point symmetries, and extended classes of equivalence transformations, is given in Section 2. The section also contains several illustrative examples, and ends with an outline of a systematic procedure of computation of Lie groups of generalized equivalence transformations, closely related to the algorithm of point symmetry computation. Section 3 presents a typical sequence of steps for complete symbolic computation of Lie groups of generalized equivalence transformations using the `GeM` `Maple` package.

Section 4 is devoted to examples of symbolic computation of equivalence transformations of PDE models. Detailed `Maple` code is included for each example. In the first example, equivalence transformations are computed for the well-known three-parameter family of dimensional Korteweg–de Vries (KdV) equations, and are used to invertibly reduce the family to a single canonical KdV equation involving no arbitrary parameters. In the second example, we compute nonlocal equivalence transformations of a family of nonlinear wave equations. In particular, nonlocal equivalence transformations are found as point transformations of a potential system, and do not correspond to local equivalence transformations of the model.

The final example of Section 4 is devoted to the computation of a Lie group of generalized equivalence transformations of the KdV–Burgers PDE family; it is demonstrated that the transformation component corresponding to the dependent variable explicitly depends on an arbitrary element that is an *arbitrary function* of the model. This is the first computational example of this situation in the literature (see also Remark 2.6).

An additional example considered in Section 5 presents a recent model of finite anti-plane shear displacements in nonlinear incompressible hyperelastic fiber-reinforced hyperelastic materials [24]. Such models arise in the description of multiple types of nonlinear media, including biological membranes [25]. The systematically computed local equivalence transformations are used to drastically simplify the model, by invertible transformations, through the complete elimination of arbitrary elements. Together with the classical example of the Korteweg–de Vries model in Section 4, the example of Section 5 addresses the questions of primary importance for the analysis of nonlinear equations arising in physics and applied science—the identification of essential parameters of the model, and the possibility of invertible parameter reduction.

Section 6 concludes the paper with a discussion of the presented algorithms, their efficiency, results, related classification problems, and an overview of open questions and possible work directions in the area.

Throughout the paper, summation in repeated indices is assumed where appropriate.

2. Equivalence transformations, their computation, and extensions

2.1. Lie groups of equivalence transformations

Consider a family $\mathcal{F}_K$ of DE systems $\mathbf{R}\{x\,; u; K\}$:

$$R^\sigma(x, u, \partial u, \ldots, \partial^k u, K) = 0, \quad \sigma = 1, \ldots, N, \tag{2.1}$$

involving n independent variables $x = (x^1, \ldots, x^n)$, m dependent variables $u(x) = (u^1(x), \ldots, u^m(x))$, and L arbitrary elements (constitutive functions and/or parameters) $K = (K^1, \ldots, K^L)$. In (2.1), partial derivatives are denoted by $u^\mu_i = \partial u^\mu(x)/\partial x^i$. The symbol

$$\begin{aligned}\partial^p u &= \left\{u^\mu_{i_1\ldots i_p} \mid \mu = 1, \ldots, m;\;\; i_1, \ldots, i_p = 1, \ldots, n\right\}\\ &= \left\{\frac{\partial^p u^\mu(x)}{\partial x^{i_1}\ldots\partial x^{i_p}} \mid \mu = 1, \ldots, m; i_1, \ldots, i_p = 1, \ldots, n\right\}\end{aligned}$$

is used to denote the set of all partial derivatives of order p, $p = 1, 2, \ldots$. The arbitrary elements K in (2.1) may be either arbitrary constants or arbitrary functions within some class. Constitutive functions are assumed to be sufficiently smooth functions of dependent and/or independent variables and/or derivatives of the dependent variables, or combinations thereof.