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Stochastic finite element response analysis using random eigenfunction expansion

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ABSTRACT

A mathematical form for the response of the stochastic finite element analysis of elliptical partial differential equations has been established through summing products of random scalars and random vectors. The method is based upon the eigendecomposition of a system's stiffness matrix. The computational reduction is achieved by only summing the dominant terms and by approximating the random eigenvalues and the random eigenvectors. An error analysis has been conducted to investigate the effect of the truncation and the approximations. Consequently, a novel error minimisation technique has been applied through the Galerkin error minimisation approach. This has been implemented by utilising the orthogonal nature of the random eigenvectors. The proposed method is used to solve three numerical examples: the bending of a stochastic beam, the flow through a porous media with stochastic permeability and the bending of a stochastic plate. The results obtained through the proposed random eigenfunction expansion approach are compared with those obtained by using direct Monte Carlo Simulations and by using polynomial chaos.

1. Introduction

Uncertainties can substantially affect the analysis of physical structures. These uncertainties can occur in the properties of the material, in the geometry or boundary conditions of the structure or in the applied loads [1]. In order to represent the uncertainties that occur in physical systems, a stochastic finite element method [SFEM] can be applied. This method has been applied to numerous problems including structural mechanics, fluid mechanics and heat transfer problems. Both static [2,3] and dynamic [4,5] scenarios can be represented through this method. In this work, a stochastic elliptic partial differential equation is considered

$$-\nabla^n[a(x,\omega)\nabla^n u(x,\omega)] = p(x) \qquad x \text{ in } \mathcal{D} \tag{1}$$

with the associated Dirichlet condition

$$u(x,\omega) = 0; \qquad x \text{ on } \mathcal{D} \tag{2}$$

In Eq. (1), u refers to the governing variable and ∇ refers to the differential operator (for a single dimensional problem $\nabla = \frac{\partial}{\partial x}$) and $n = 1, 2$. The value of n would depend on the physical problem under consideration. When dealing with a flow through a porous media n would be equal to 1, and for the bending of a beam or a plate n would be equal to 2. Both scenarios are discussed in this paper. The spatial dimension under consideration is a bounded domain $\mathcal{D} \in \mathbb{R}^d$ with piecewise Lipschitz boundary $\partial\mathcal{D}$ where d is less than four. $(\Omega, \mathcal{F}, \mathcal{D})$ is a probability space where $\omega \in \Omega$ is a sample point from the sampling space Ω, $\mathcal{F}$ is the complete σ-algebra over the subsets of Ω and P is the probability measure. In Eq. (1) $a : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a random field [6], which can be viewed as a set of random variables indexed by $x \in \mathbb{R}^d$. We assume the random field $a(x,\omega)$ to be stationary, square integrable and non-negative. Following the discretization of Eq. (1) through the SFEM [7], this work aims to produce a new solution approach through the use of random eigenfuncation.

Direct Monte Carlo Simulation [MCS] has been widely used in collaboration with the SFEM [8]. Although this is a relatively simple method, using a large number of realisations in conjunction with high dimensional matrices can make this method computationally expensive. Numerous approaches have been proposed in order to reduce the computational time. Multi-level Monte Carlo is one such method where the variance of the Monte Carlo estimator is reduced [9–11]. Other accelerating methods include centroidal Voronoi tessellations [12,13], Latin hypercube sampling [14] and quasi Monte Carlo [15,16]. In spite of the high computational cost linked with the direct MCS method, the error and the computational cost associated with other methods are regularly compared with the direct MCS method [17,18].

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Other methods are available to calculate functional statistics which avoids the use of computationally expensive sampling methods. One such approach is the perturbation method [19–21]. In such an approach, a Taylor series expansion is used to approximate the structural response. By assuming that the Taylor series converges, the greater the number of terms kept in the series, the higher the accuracy of the response [22]; however publications using an order greater than two are uncommon due to high computational cost. A considerable disadvantage is that the coefficients of variation can't exceed 15% of the mean value of the variable under consideration [8]. Other approaches include Neumann expansions [23–25] and linear algebra techniques [26]. The stochastic Galerkin method is also popular [27,28]. This method projects the response on an orthogonal basis that spans the stochastic space.

Another class of methods which have been widely used are spectral methods. This class originates from Wiener's work [29] where the homogeneous chaos method is initially defined. One of the first applications of the chaos expansion for stochastic finite elements is contained in [7]. If the random variables are deemed Gaussian, a polynomial chaos approach can be considered. This approach has been widely used to model different physical scenarios including structural [7], flow [30] and heat transfer [31] problems. However, due to the high computational cost of large systems, numerous reduction methods have been suggested. These include [32] where a spectral decomposition of the deterministic matrix is performed, and only the dominant eigenvalues and eigenvectors retained. [33] have designed an optimisation algorithm which makes the polynomial chaos approximation computationally feasible. Other spectral methods include the Wiener-Askey chaos expansion [34,35] and the reduced basis method [36,37].

In Section 2 an overview of the spectral stochastic finite element method is presented. The random eigenfunction approach is proposed in Section 3, whilst Section 4 discusses different ways of approximating random eigenvalues and eigenvectors. Section 5 includes a novel error analysis which is followed by a novel error minimising technique. The new approach is applied to a stochastic Euler-Bernoulli beam, a flow through a stochastic porous media and to the stochastic mechanics of a bending elastic plate in Section 6 and the major conclusions are presented in Section 7.

2. Discretization of the stochastic PDE

The random process $a(x,\omega)$ seen in Eq. (1) can be expanded by a generalised Fourier expansion known as the Karhunen-Loève expansion

$$a(x,\omega) = a_0(x) + \sum_{i=1}^{\infty} \sqrt{\tilde{\lambda}_i}\,\tilde{\xi}_i(\omega)\tilde{\boldsymbol{\phi}}_i(x) \tag{3}$$

Here a_0 is the mean function and $\tilde{\lambda}_i$ and $\tilde{\boldsymbol{\phi}}_i(x)$ are the eigenvalues and eigenvectors that satisfy the integral equation

$$\int_D C_a(x_1,x_2)\tilde{\boldsymbol{\phi}}_j(x_1)\,dx_1 = \tilde{\lambda}_j\tilde{\boldsymbol{\phi}}_j(x_2) \qquad \forall\, j = 1,2,\ldots \tag{4}$$

where $C_a(x_1,x_2)$ is the covariance function. The $\tilde{\xi}_i(\omega)$ seen in the Karhunen-Loève expansion corresponds to random variables. If the random process is deemed Gaussian, $\tilde{\xi}_i(\omega)$ would be standard Gaussian random variables. For other types of random processes, the random variables may possess other distribution types. After truncating the series seen in Eq. (3) to the Mth term, the resulting equation can be substituted into the original stochastic elliptical partial differential equation. By applying appropriate boundary conditions, the discretized equation takes the form

$$\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\omega)\mathbf{A}_i\right]\mathbf{u}(\omega) = \mathbf{f} \tag{5}$$

where $\mathbf{A}_0 \in \mathbb{R}^{n\times n}$ represents a deterministic, positive definite, symmetric matrix. $\mathbf{A}_i \in \mathbb{R}^{n\times n}$ are general symmetric matrices for $i = 1,2,\ldots M$, $\mathbf{u}(\omega) \in \mathbb{R}^n$ the response vector and $\mathbf{f} \in \mathbb{R}^n$ the deterministic input force vector. The details of obtaining the discretized equivalent of Eq. (1) have been omitted, but can be located in numerous textbooks including [7]. The method proposed in this paper is general in nature, therefore the random variables seen in Eq. (5) are not restricted to any specific distribution.

3. Random eigenfunction expansion

3.1. Motivation behind the proposed approach

For simplicity, we express Eq. (5) as

$$\mathbf{A}(\omega)\mathbf{u}(\omega) = \mathbf{f} \tag{6}$$

where the random matrix $\mathbf{A}(\omega) = \mathbf{A}_0 + \sum_{i=1}^{M}\xi_i(\omega)\mathbf{A}_i$. The matrix $\mathbf{A}(\omega)$ can be considered as a random stiffness matrix. As the system under consideration is static, a mass matrix is not required. We will consider problems where the value of M and the number of degrees of freedom in a system are sufficiently large. For small values of M computational reduction can be achieved. However when M is sufficiently large the solution of Eq. (6) poses computational challenges.

The exact solution to the set of stochastic linear equations given above can be obtained through direct MCS. Convergence is guaranteed if the number of realisations is sufficiently large and all realisations of $\mathbf{A}(\omega)$ are positive definite. However, direct MCS can be seen as a computationally expensive method [38], especially if there is a large number of stochastic linear equations to be solved. In order to avoid the use of direct MCS, alternative methods have been explored. The response of Eq. (6) can be represented through summing products of random scalars and deterministic vectors

$$\mathbf{u}(\omega) = \sum_{j=1}^{M_1} a_j(\omega)\mathbf{g}_j \tag{7}$$

where $a_j(\omega) \in \mathbb{R}$ and $\mathbf{g}_j \in \mathbb{R}^n$ represent the random scalars and deterministic vectors respectively. M_1 corresponds to the number of terms in the summation. Eq. (7) can be considered as the polynomial chaos method

$$\mathbf{u}(\omega) = \sum_{k=1}^{P} H_k(\xi(\omega))\mathbf{u}_k \tag{8}$$

where $H_k(\xi(\omega))$ represents the polynomial chaoses (corresponding to the random scalars), and $\mathbf{u}_k$ represents unknown deterministic vectors that need to be determined. The value of P is determined by a basic random variable M and by the order of the Polynomial Chaos expansion p. In this instance, M corresponds to the order of the Karhunen-Loève expansion. The value of P is determined by the following expression

$$P = \sum_{j=0}^{p} \frac{(M+j-1)!}{j!(M-1)!} \tag{9}$$

It is evident that P increases rapidly when either the order of the Karhunen-Loève expansion or the order of the Polynomial Chaos expansion is increased. The unknown vector $\mathbf{u}_k$ can be obtained by using a Galerkin error minimising approach [7]. This approach leads to a system of linear equations of size $nP \times nP$. A possible drawback to this approach is the high computational cost if either n or P is large.