Engineering Analysis with Boundary Elements 85 (2017) 70–86

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Analytical derivation and numerical experiment of degenerate scale by using the degenerate kernel of the bipolar coordinates

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ARTICLE INFO

Keywords:
Degenerate scale
Degenerate kernel
Bipolar coordinates
Eccentric annulus

ABSTRACT

Degenerate scales of an eccentric annulus and an infinite plane with two identical circular holes in the boundary integral equation method (BIEM) are analytically derived and numerically implemented in this paper. To analytically study the degenerate scale of the BIE, the closed-form fundamental solution of the two-dimensional Laplace equation, $\ln r$, is expanded by a degenerate (separate) kernel in terms of the bipolar coordinates. It is proved that unit radius of the outer circle dominates the degenerate scale of eccentric annulus. An analytical formula of degenerate scale for the infinite plane with two identical circular boundaries was also derived at the first time. In addition, null fields of the domain and complementary domain for the ordinary and degenerate scales are both shown, respectively. Finally, comparison with available results and the BEM data are well done.

1. Introduction

Boundary integral equation method (BIEM)/boundary element method (BEM) is efficient and accurate for solving two-dimensional problems governed by the Laplace equation, the Navier equation or the biharmonic equation. However, the Dirichlet type problem of a special domain may yield a nonunique solution once the single-layer kernel is used. This special size is called the degenerate scale. The degenerate scale is related to Gamma contour [1], logarithmic capacity [2], critical value [3] and transfinite boundary [4] but the concepts are the same. Therefore, how to predict the degenerate scale appearing in the BIEM/BEM is very important and is not trivial in the development of the BEM. For this reason, many researchers paid attention to this issue in recent years [5–7].

When the real size is at a degenerate scale in the BIEM/BEM implementation, it results in a singular influence matrix due to a weakly singular kernel (U). In other words, the occurrence of the degenerate scale is inherent in the integral equation. Hence, the degenerate scale is not physically realizable but is mathematically interpretable. From the viewpoint of mathematics, there are two ways to understand the degenerate scale. One is the non-uniqueness solution in the BIEM/BEM. The other is the unit logarithmic capacity corresponding to the conformal radius in the complex analysis [8]. When the logarithmic capacity is equal to 1, the corresponding scale is a degenerate one. There were analytical formulae of the logarithmic capacity for the fourteen geometry shapes presented by Landkof [9]. Dijkstra [10] numerically examined the value of the degenerate scale for some geometry shapes [6] by using the BEM. However, the derivation of the analytical formulae was not given in Landkof's book [9]. Later, Rumely [2] employed the conformal mapping to analytically derive the logarithmic capacity for many shapes such as the circle, the ellipse and the disjoint circles. However, the relation between the logarithmic capacity and the degenerate scale was not addressed in their books [2,9]. Until 2013, Kuo et al. [11] employed the Riemann conformal mapping to link the unit logarithmic capacity and the degenerate scale. One way to understand the degenerate scale which results in the non-uniqueness solution in the BIEM/BEM is the range deficiency of the integral operator of a weakly singular kernel, $\ln r$.

Chen et al. [12] employed the degenerate kernel and the circulant to analytically study the degenerate scale of circular and annular geometries in the continuous and discrete systems, respectively. Later, the numerical demonstration was achieved in an eccentric annulus case by Chen and Shen [13]. But the analytical expressions were not provided. An annular region has also been considered for the harmonic equation by He et al. [14], Liu and Lean [15]. Possible degenerate scales were studied in both continuous and discrete systems. However, the

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http://dx.doi.org/10.1016/j.enganabound.2017.08.006
Received 2 December 2016; Received in revised form 3 August 2017; Accepted 3 August 2017
0955-7997/

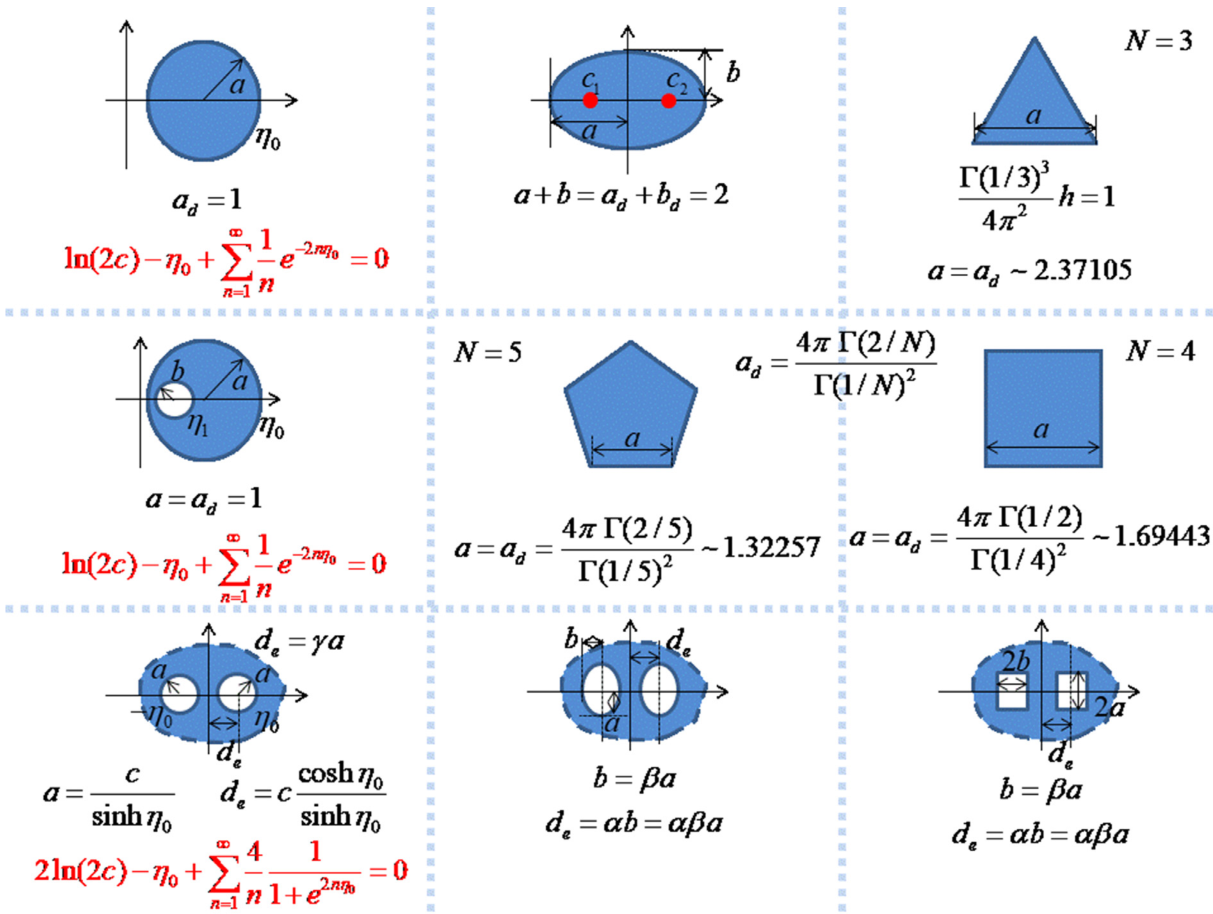

Fig. 1. Degenerate scale of several shapes of domain.

circulant property fails in the non-circular or non-annular case. Therefore, Chen et al. [16] used the BEM to numerically investigate the degenerate scale of an ellipse and found that there exist two degenerate scales in the plane elasticity. Related works were done by Chen and his coworkers [17–19]. Besides, Chen et al. [20] also extended the BIEM/BEM to study the degenerate scale for a circular thin plate (biharmonic equation). Regarding the multiply-connected domain, it is found that the degenerate scale depends on the outer boundary contour [21,22]. Chen and Shen [13] also numerically demonstrated that the degenerate scale of an eccentric annulus depends on the unit radius of the outer circle. However, no analytical proof was done to the best of authors' knowledge. Here, we would present it by using the bipolar coordinates for an eccentric annulus.

Corfdir and Bonnet [23] studied the Laplace problem of degenerate scale for a half-plane domain. In their paper, they claimed that the degenerate scale depends on the type of the boundary condition on the line bounding the half-plane. The degenerate scale only occurs for the Neumann boundary condition. On the contrary, it does not appear for the Dirichlet type. The half-plane problem can be transformed into an infinite-plane problem with the symmetric or the anti-symmetric Dirichlet boundary condition. Later, Chen [24] employed a null-field BIEM to study the same problem. Numerical results [24] also support the finding in [23]. However, from the viewpoint of using the indirect BIEM/BEM, the influence matrices constructed by the weakly singular kernel (ln*r*) for those two infinite-plane problems are the same. For two holes in the infinite plane, they may have the same degenerate scale no matter that it is symmetric or anti-symmetric Dirichlet boundary conditions since the influence matrices are the same. In those two papers [23, 24], both of them employed the image method to construct the corresponding Green's function. The boundary condition on the line bounding the half-plane can be satisfied in advance by using the Green's function in their BEM formulations. In this way, the degenerate scale is free for the Dirichlet condition on the line bounding the half-plane. In this paper, we adopted the usual kernel (ln*r*) in our BEM formulation for all kinds of boundary conditions. This is the reason why we will examine the present result and those of the two papers. This finding also verifies again that the degenerate scale depends on the kernel function in the BEM/BIEM.

In addition, the null and nonzero fields for ordinary and degenerate scales are also the main focus of the present study. Chen et al. [25] analytically studied the field of both interior and exterior domains for an elliptical case. They found that the trivial boundary potential may result in a nontrivial boundary flux when the geometric size is at a degenerate scale. It means that the fields inside and outside the domain are null and nonzero for a domain at the degenerate scale, respectively. This phenomenon is not physically realizable and is opposite to the phenomenon of ordinary scale. In 2012, Chen et al. [26] employed the null-field BIEM in conjunction with the degenerate kernel to revisit the same problem. The same result was obtained. Later, Kuo et al. [6] used the BEM to numerically examine the null and nonzero fields for regular N-gon domains including right triangle, square, regular 5-gon and regular 6-gon. All numerical phenomena about degenerate scales obtained by the BEM agreed with the analytical prediction. Furthermore, true and spurious eigensolutions also have similar behavior of null and nonzero modes. Chen et al. [27] employed both BEM and the null-field BIEM to numerically demonstrate and analytically examine the null and nonzero modes for circular, elliptical, annular and confocal elliptical membranes. Not only true eigenmodes but also spurious eigenmodes were addressed. However, all the studies of degenerate scale and spurious eigenmodes