Engineering Analysis with Boundary Elements 82 (2017) 104–110

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A boundary element method recursive procedure applied to Poisson's problems

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ARTICLE INFO

Keywords:
Boundary element method
Recursive procedure
Weighted residual method
Poisson's equation

ABSTRACT

This paper describes a simple procedure to increase the accuracy of the boundary element method (BEM) results in Poisson's problems using coarse meshes. Usually, BEM values at internal points are obtained by reusing the boundary integral equation, after having calculated all variables at the nodal points on the boundary. Accuracy in results of these internal points is superior to that obtained at boundary nodes and the reason for that can be assigned to a new minimization of residuals performed. Therefore, this idea can be used to improve BEM results by means of choosing new source points on the boundary at positions different from those of the original nodes. Tests carried out with problems governed by Laplace's equation and Navier's equation were successful; thus, this procedure is now applied to Poisson's problems that allow a more comprehensive evaluation of the performance of proposed technique.

1. Introduction

Numerical results show that the BEM accuracy for internal points is better than that for boundary nodal points. This performance can be justified considering mathematical fundamentals of the weighted residual method (WRM) [1], from which the main discrete numerical methods, such as the finite element method [2], the finite difference method [3] and the BEM can be derived and also interpreted. Since values of internal variables are calculated by reusing the boundary integral equation, this procedure can be understood as equivalent to a new residuals minimization. The main difference is that the new weighted residual integral equation contains all nodal variables previously calculated.

Thus, the idea of recursive procedure is simply to apply again the boundary integral sentence putting the source points on the boundary in different positions from those of the original source points. Naturally, the procedure is not applied at points at which the basic variables are prescribed. The expectancy is to achieve results with better accuracy, as occurs for internal points.

Indeed, the recursive procedure has already been successfully applied to Laplace's problems [4] and also in linear elastic problems, governed by the Navier's equation [5]. Interesting results were obtained in these applications, particularly for derivatives of basic variable, such as flux in potential problems and tractions in elasticity. In this work, the technique is applied to Poisson's problems, in which a much broader number of examples can be solved, allowing better assessment of the technique in scalar problems. Recursive results are compared with direct nodal results, that is, results obtained by solution of standard BEM matrix equation system; both results are compared with reliable solutions, being analytical or discrete, taken as reference for better evaluation.

It is important to highlight one aspect in Poisson's problems which must be carefully considered: the mathematical treatment of the integral related to the source term. Avoiding errors relating to approximation of domain integral, problems that are adequately solved by the Galerkin tensor technique [6] were chosen so that errors related to approximation of domain integrals were avoided. Numerical experiences using the dual reciprocity boundary element method [7] were not successful, probably due to the interference of the radial basis interpolation functions [8,9].

2. Poisson's equation

Consider a two-dimensional homogeneous isotropic body subjected to a thermal or mechanical field at a steady state and that there are sources, sinks or external actions that act directly in the field. Denoting by $u_o(X)$ the potential scalar variable which represents the field, the differential equation associated with this problem, the Poisson's equation,

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http://dx.doi.org/10.1016/j.enganabound.2017.06.003
Received 26 September 2016; Received in revised form 31 May 2017; Accepted 12 June 2017
0955-7997/

in indicial notation is given by:

$$u_o(X)_{,ii} = p(X) \tag{1}$$

The essential and natural boundary conditions are defined respectively by the following equations:

$$u_o(X) = \bar{u}(X) \quad \text{on} \quad \Gamma_u; \text{ and } \quad u_{o,i}n_i(X) = \bar{q}(X) \quad \text{on} \quad \Gamma_q \tag{2}$$

In Eq. (2) n_i is the external unitary normal vector on the boundary and $\Gamma = \Gamma_u + \Gamma_q$.

3. Weighted residuals boundary integral equation

Considering the potential $u(X)$ and flux $q(X)$ as approximate solutions for the exact values, the WRM associated with the Poisson equation is given by the following integral equation [10]:

$$\int_{\Gamma_u} [\bar{u}(X) - u(X)]q^*(\xi; X)d\Gamma + \int_{\Gamma_q} [q(X) - \bar{q}(X)]u^*(\xi; X)d\Gamma = \int_{\Omega} [p(X) - u_{,ii}(X)]u^*(\xi; X)d\Omega \tag{3}$$

In Eq. (3), weighting functions $u^*(\xi, X)$ are the fundamental solution of the Poisson problem and $q^*(\xi, X)$ is its normal derivative [1,6], whose analytical formulas are well-known. Eq. (3) includes minimization of all types of residuals that may appear on the numerical solution, either in the field $\Omega(X)$ or on the $\Gamma_u(X)$ and $\Gamma_q(X)$ borders.

As the fundamental solution and its normal derivatives are associated to a correlate physical problem, considering the mathematical principles of the WRM, an efficient functional space is constructed, in order to minimize numerical residuals accurately.

Well known mathematical procedures [6] lead to an integral equation, which is more suitable for application of classic BEM discretization procedure:

$$c(\xi)u(\xi) + \int_{\Gamma} u(X)q^*(\xi; X)d\Gamma - \int_{\Gamma} q(X)u^*(\xi; X)d\Gamma = \int_{\Omega} p(X)u^*(\xi; X)d\Omega \tag{4}$$

The coefficient $c(\xi)$ depends on the position of the source point ξ with respect to the physical domain $\Omega(X)$. For problems in which the domain actions are harmonic, the technique that uses the Galerkin tensor $G^*(\xi;X)$ is advantageous. Thus, except for discretization errors, the right side of Eq. (4) is integrated exactly, without residuals minimization with respect the function $p(X)$. In this case, one has:

$$\int_{\Omega} p(X)u^*(\xi; X)d\Omega = \int_{\Omega} p(X)G^*_{,ii}(\xi; X)d\Omega = \int_{\Gamma} p(X)G^*_{,i}(\xi; X)n_i(X)d\Gamma - \int_{\Gamma} p_{,i}(X)n_i(X)G^*(\xi; X)d\Gamma \tag{5}$$

The following functions appear in the kernels of the right hand side of Eq. (5):

$$G^*(\xi; X) = \frac{1}{8\pi} r^2(1 - \ln r) \tag{6}$$

$$G^*_{,i}(\xi; X)n_i(X) = \frac{1}{8\pi}\left[r + 2\ln\frac{1}{r}\right]\frac{\partial r}{\partial x_i} \tag{7}$$

4. Recursive procedure

The idea of the recursive procedure is to reapply the integral equation after the solution of the standard BEM system, using new source points. It is well known that the BEM collocation usually considers boundary nodal points as source points. Thus, residuals are null at boundary nodes; however, in such points the error is not null, not even minimum. Thus, the aim here is to evaluate the capability of the recursive procedure to make residual minimization more effective. This is not similar to the mesh refinement strategy, in which the reduction of numerical error is due to the addition of a large number of collocation points.

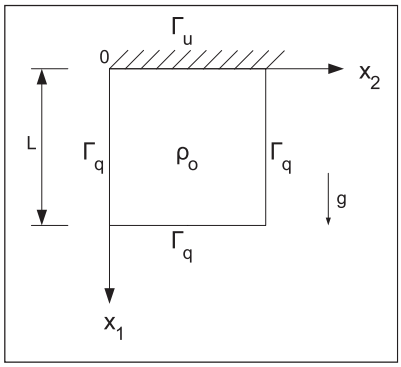

Fig. 1. Rod submitted to constant domain force.

The recursive boundary integral equation procedure is closely related to the collocation BEM adaptive techniques, since in the absence of a better criterion of error estimation, a new integral equation is generated for new source points. However, despite the similarities, there is a conceptual difference. Neglecting the sense of error minimization, the adaptive techniques only search a factor to indicate regions more suitable to the mesh refinement. Differently, the recursive procedure indicates that when the integral residual sentence is applied again a reduction of the errors at the new collocation points can be achieved.

Regarding operational features, all nodal values of potential and normal derivatives are available, such as occurs in calculation at internal points. There is no restriction to put the new source points on the boundary again. It must be avoided to choose original boundary nodes for redundancy. It is also strategic to choose the new source points equidistant of the boundary nodes. Thus, using linear elements as is done in this work, recursive points are centered on the boundary elements and the discretized form of boundary equations are relatively simple.

Considering the letter "e" referring to the boundary element; the known nodal variables being U^e, Q^e, P^e and Z^e; the index k referring to the nodal points and $\phi(X)$ being the shape function, potential value at a recursive point ξ^r is given by:

$$(0.5)u(\xi^r) = \sum_{e=1}^{N} Q^e_k \int_{\Gamma_e} \phi_k(X)u^*(\xi^r; X)d\Gamma_e - \sum_{e=1}^{N} U^e_k \int_{\Gamma_e} \phi_k(X)q^*(\xi^r; X)d\Gamma_e + \sum_{e=1}^{N} P^e_k \int_{\Gamma_e} \phi_k(X)G^*_{,i}(\xi^r; X)n_i(X)d\Gamma_e - \sum_{e=1}^{N} Z^e_k \int_{\Gamma_e} \phi_k(X)G^*(\xi^r; X)d\Gamma_e \tag{8}$$

Eq. (8) was taken, for sake of simplicity:

$$p_{,i}(X)n_i(X) = \frac{\partial p}{\partial x_1}\frac{\partial x_1}{\partial n} + \frac{\partial p}{\partial x_2}\frac{\partial x_2}{\partial n} = z_1 n_1 + z_2 n_2 = Z \tag{9}$$

The calculation of normal derivative of potential requires special attention, since one of the kernels of the integral recursive equation presents a hyper-singular behavior [11,12]. However, using the recursive procedure the new source points are located between nodal points, thus defined along a smooth curve, which greatly simplifies the treatment of these integrals. Then, the boundary integral equation for normal