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Non-Euclidean distance fundamental solution of Hausdorff derivative partial differential equations

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A B S T R A C T

The Hausdorff derivative partial differential equations have in recent years been found to be capable of describing complex mechanics and physics behaviors such as anomalous diffusion, creep and relaxation in fractal media. But most research is concerned with time Hausdorff derivative models, and little has been reported on the numerical solution of spatial Hausdorff derivative partial differential equations. In this study, we derive the fundamental solutions of the Hausdorff derivative Laplace, Helmholtz, modified Hemholtz, and convection-diffusion equations via a non-Euclidean metric, called the Hausdorff fractal distance. And then the singular boundary method is used to numerically simulate the steady heat transfer governed by the Hausdorff Laplace equation in comparison with the corresponding fractional Laplacian models. Numerical experiments show the validity and applicability of the derived fundamental solution of the Hausdorff Laplace equation.

1. Introduction

Fractal has attracted great attention in many diverse fields since its inception. However, its success in real-world applications is largely restricted because its corresponding calculus formalism is not fully established. Nowadays there exist a few calculus methodologies for modeling physics behaviors in fractal media. The analysis on fractals is a classical method based on the fractal sets, and is mathematically sophisticated and difficult to use in practice. Thus, this strategy does not attract a lot of attention in scientific and engineering communities [1].

In recent decades, the fractional calculus has become very popular in modeling anomalous diffusion, creep, relaxation, and power law viscous dissipation, just to mention a few [2–8]. The fractional calculus is in fact an integro-differential operator and can inherently well describe some non-local and history-dependent behaviors, but this methodology encounters computationally very expensive costs due to its non-local property. In addition, the fractional derivative diffusion models are restricted to describe the problems underlying the Levy and the ML statistics, and its underlying relationship with the fractal was not clear [9] and was somewhat revealed very recently in [10].

As an alternative modeling approach to the fractional calculus, the Hausdorff derivative was introduced [11] as a local operator to overcome high computing costs of the non-local fractional derivative. In addition, the Hausdorff derivative is capable to describe the diffusion problems underlying the well-known stretched Gaussian statistics and the Kohlrausch–Williams–Watts stretched exponential decay [11]. It is noted that the Hausdorff derivative order has clear physical meaning and is directly related to the fractal dimension [10]. As pointed out in Ref. [12], the Hausdorff derivative is in fact equivalent to the fractal derivative proposed in Refs. [13, 14], and its rigorous mathematical analysis is provided in Ref. [14]. In recent years, the Hausdorff derivative has attracted great attention and is widely used in various complex problems in scientific and engineering fields, such as water resources [15], anomalous diffusion [16], the creep of viscoelastic materials [17], magnetic resonance imaging (MRI) [18,19], heat conduction [20], and economics [21].

Ref. [11] gave the fundamental solution of the one-dimensional Hausdorff derivative transient diffusion equation via the fractal metric spacetime, namely, the Hausdorff fractal distance called in this study. This paper employs this non-Euclidean metric to derive the fundamental solution of the Hausdorff derivative Laplace, Helmholtz, and convection-diffusion equations. And then the singular boundary method (SBM) [22,23], a recent meshless boundary collocation method based on the fundamental solution, is applied to numerically simulate the Hausdorff derivative Laplace equation of steady heat transfer problems. Numerical results are discussed and compared with those given in Ref. [24].

A brief outline of the rest of this paper is as follows: Section 2 reviews the Hausdorff derivative, and introduces the definition of the Hausdorff

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Table 1
Fundamental solutions for most commonly used Hausdorff differential operators on the non-Euclidean Hausdorff fractal distance of two and three topological dimensions.

Operator	Two-dimensional	Three-dimensional
Δ	$-\frac{1}{2\pi}\ln r_d^{\beta}$	$\frac{1}{4\pi r_d^{\beta}}$
$\Delta+\lambda^2$	$-\frac{i}{4}H_0^{(2)}(\lambda r_d^{\beta})$	$\frac{e^{-i\lambda r_d^{\beta}}}{4\pi r_d^{\beta}}$
$\Delta-\lambda^2$	$\frac{1}{2\pi}K_0(\lambda r_d^{\beta})$	$\frac{e^{i\lambda r_d^{\beta}}}{4\pi r_d^{\beta}}$
$D\Delta-\frac{\partial}{\partial t}$	$\frac{H(\Delta t)}{4\pi D\Delta t^{\alpha}}e^{-(r_d^{\beta})^2/4D\Delta t^{\alpha}}$	$\frac{H(\Delta t)}{(4\pi D\Delta t^{\alpha})^{3/2}}e^{-(r_d^{\beta})^2/4D\Delta t^{\alpha}}$
$D\Delta-\mathbf{v}\cdot\nabla-\frac{\partial}{\partial t}$	$\frac{H(\Delta t)}{4\pi D\Delta t^{\alpha}}e^{-(r_d^{\beta})^2/4D\Delta t^{\alpha}-\lvert\mathbf{v}\rvert^2\Delta t^{\alpha}/4D+\mathbf{v}\cdot\mathbf{r}_d^{\beta}/2D}$	$\frac{H(\Delta t)}{(4\pi D\Delta t^{\alpha})^{3/2}}e^{-(r_d^{\beta})^2/4D\Delta t^{\alpha}-\lvert\mathbf{v}\rvert^2\Delta t^{\alpha}/4D+\mathbf{v}\cdot\mathbf{r}_d^{\beta}/2D}$

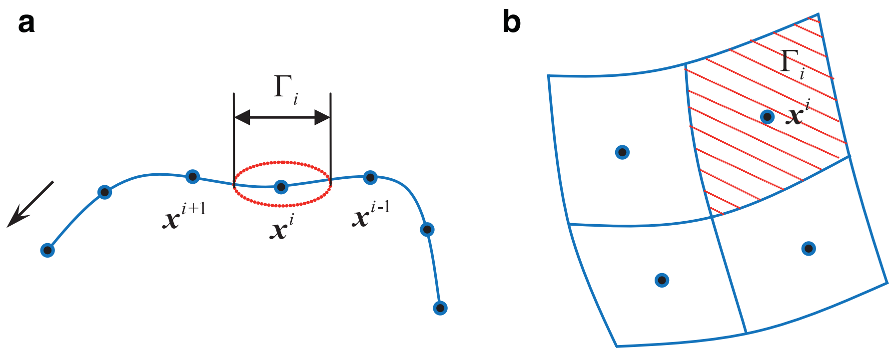

Fig. 1. Nodal integration domain in (a) 2D and (b) 3D problems.

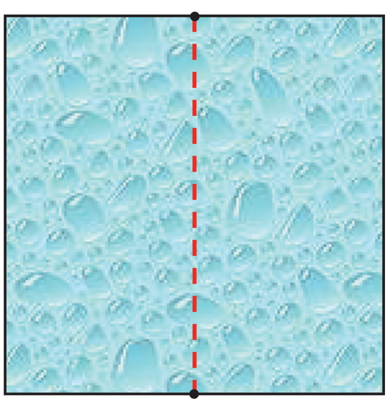

Fig. 2. Steady heat transfer in a 2D fractal media.

Table 2
Errors and CPU time by using the SBM and the BEM with different number of nodes (or elements) for the two-dimensional heat transfer problem of integer dimension.

N	BEM		SBM	
	L_∞	CPU time(s)	L_∞	CPU time(s)
200	2.0814×10^{-2}	0.60840	3.0918×10^{-2}	0.15600
400	3.9964×10^{-3}	2.26200	5.3899×10^{-3}	0.70200
600	1.4664×10^{-3}	5.44440	1.9052×10^{-3}	2.44920
800	7.1454×10^{-4}	12.1681	9.0748×10^{-4}	6.89520
1000	4.0717×10^{-4}	19.9213	5.0949×10^{-4}	11.1853

fractal distance. And then Section 3 gives the fundamental solutions of a few typical partial differential equation models. Section 4 discusses numerical experiment results of steady heat conduction problems governed by the Hausdorff Laplace equation via the SBM. Finally, some conclusions are drawn in Section 5.

2. Hausdorff derivative and non-Euclidean Hausdorff fractal distance

Considering a particle movement in terms of fractal time, the movement distance can be calculated by [10]

$$l(\tau)=v\left(\tau-t_0\right)^{\alpha}, \tag{1}$$

where l denotes the distance, v represents the uniform velocity, τ the current time instance, t_0 the initial instance, α the fractal dimensionality in time. When velocity varies with time, the Hausdorff integral distance is given by

$$l(t)=\int_{t_0}^{t} v(\tau)d\left(\tau-t_0\right)^{\alpha}. \tag{2}$$

We can derive the Hausdorff derivative from the above Hausdorff derivative expression (2)

$$\frac{dl}{dt^{\alpha}}=\lim_{t'\to t}\frac{l(t)-l(t')}{\left(t-t_0\right)^{\alpha}-\left(t'-t_0\right)^{\alpha}}=\frac{1}{\alpha\left(t-t_0\right)^{\alpha-1}}\frac{dl}{dt}. \tag{3}$$

Let the initial instance t_0 be set zero, we have [11]

$$\frac{dl}{dt^{\alpha}}=\lim_{t'\to t}\frac{l(t)-l\left(t'\right)}{t^{\alpha}-t'^{\alpha}}=\frac{1}{\alpha t^{\alpha-1}}\frac{dl}{dt}. \tag{4}$$

By analogy with the above time Hausdorff derivative, the Hausdorff derivative in space is stated as

$$\frac{du}{dx^{\beta}}=\lim_{x'\to x}\frac{u(x)-u(x')}{x^{\beta}-x'^{\beta}}=\frac{1}{\beta x^{\beta-1}}\frac{du}{dx}, \tag{5}$$

where β represents the Hausdorff fractal in space. Note that the origin of spatial coordinate system in expression (5) is assumed zero.

The first author of this paper introduced the concept of the fractal metric spacetime in one-dimensional topological fractal media [11]

$$\begin{cases}\Delta\hat{t}=\Delta t^{\alpha}\\ \Delta\hat{x}=\Delta x^{\beta}\end{cases}. \tag{6}$$

The above metric transform (6) is based on the two hypotheses: fractal invariance and fractal equivalence. In order to distinguish the above fractal metric from the fractal metric proposed by Balankin and Elizarraraz [25], which is more general and includes the above metric (6) as a special case, this paper calls the metric (6) as the Huasdorff