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Analysis of meshless weak and strong formulations for boundary value problems

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A R T I C L E I N F O

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A B S T R A C T

This paper introduces a weak meshless procedure combined with a multi-resolution numerical integration and its comparison with a strong local meshless formulation for approximating displacement and strain modeled in the form of Elliptic Boundary Value Problems (EBVPs) in one- and two-dimensional spaces. Assets and losses of both strong and weak meshless approaches are considered in detail. The meshless weak formulation considered in the current paper is the well-known Element Free Galerkin (EFG) method whereas the Local Radial Basis Functions Collocation Method (LRBFCM) is taken as a strong formulation. First aspect of the current work is implementation of the new numerical integration techniques introduced in Siraj-ul-Islam et al. (2010) and Aziz et al. (2011) [1,2] in the EFG method and its comparison with numerical integration based on standard Gaussian quadrature, adaptive integration and stabilized nodal integration techniques used in the context of EFG and other allied weak meshless formulations. Second aspect of the current work is analysis of comparative performance of the localized versions of strong and weak meshless formulations. Standard numerical tests are conducted to validate performance of both the approaches.

1. Introduction

Meshless methods have been emerged alternate simulation tools to the well-established numerical methods, such as Finite-Difference Method (FDM), Finite Element Method (FEM) and Boundary Element Method (BEM). Distinctions of the meshless methods versus the conventional methods are their independence from priori nodal connectivity or geometric meshes and ease of implementation in higher dimensions. History of meshless algorithms dates back to the Smooth Particle Hydrodynamics (SPH) method used in modelling and simulation of astrophysical phenomena [3]. After the publication of the Diffuse Element Method (DEM) [4], research interests escalated quickly towards development of different types of meshless methods. Consequently, several variants of the meshless procedures were introduced in the literature. These include Element Free Galerkin (EFG) method [5], Reproducing Kernel Particle Method (RKPM) [6], h_p-cloud method [7], Partition of Unity Finite Element Method (PUFEM) [8], Meshless Galerkin method using Radial Basis Function (MGRBF) [9], Meshless Local Boundary Integral Equation (MLBIE) method, Meshless Local Petrov Galerkin (MLPG) method [10–12], Singular Boundary Method (SBM) [13,14] and Meshless Collocation Method [15].

Unlike FEM, numerical integration is one of the crucial issues encountered in the weak meshless formulation. The main advantage of FEM is that the shape functions are piecewise polynomials of degree n and their $(n+1)$ th order derivatives vanish, thus leading to the exact calculation of stiffness matrix. On the contrary, shape functions in the weak form of meshless methods are non-polynomials (rational functions). The $p-$ th order derivatives of these functions grow as the step size between the nodal points decreases and so no quadrature scheme will have a pre-determined degree of precision in the evaluation of the integrals [16]. The difficulty in obtaining exact numerical integration in meshless methods is resulted from complexity of the shape functions. In each small integration region, the shape functions may have different forms. In some regions of the domain, derivatives of the shape functions may have oscillations as well. Consequently, insufficiently accurate numerical integration may cause deterioration and rank deficiency in the numerical solution [10,17].

There are two main sources of errors in the weak form of meshless methods. The first error is emanating from approximation of the shape functions and the second one is due to numerical integration. Because of sensitivity of the weak form of meshless methods to numerical integration, accuracy of the numerical integration has been in focus (see [18] and the references there in). Numerical integration plays an important role in accurate and stable numerical solution of meshless methods based on weak formulation [11].

In the literature, several ideas have been reported for accurate and stable evaluation of component integrals of the stiffness matrix. These

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include: nodal integration [19–22], background mesh integration [23], stress point integration [24], partition of unity quadrature [25,26], stabilized conforming nodal integration technique [21,27–29] and some other techniques [10,30,31,12]. However, difficulties of numerical integration in weak meshless formulation have not yet been fully resolved.

A substantive amount of work on nodal integration and stabilized nodal integration can be found in the literature. Stabilized nodal integration procedure while resolving some of the stability issues of the nodal integration, creates more problems, like high computational efforts and tuning of the stabilization parameter to the desired accuracy and convergence [28]. In addition, meshless method like the EFG needs a background mesh for the purpose of numerical integration.

Due to the inconvenience caused by numerical integration in the weak meshless formulation, a mathematically simpler strong form of meshless collocation approach exists in the literature [32–35]. This approach is implemented both in global and local meshless collocation forms. Global meshless collocation method is prone to ill-conditioned system matrix and is strongly dependent on the right selection of shape parameter (in the case of shape parameter dependent RBFs). The local meshless collocation methods instead are implemented on a number of local sub-domains yielding a local approximation with exponential convergence [28]. In comparison to the global meshless methods, the local meshless method produces a relatively well-conditioned system matrix. A Radial Point Interpolation Collocation Method (RPICM) with Thin Plate Spline (TPS) radial basis functions has been used for the numerical solution of nonlinear Poisson's problems [32].

Both weak and strong meshless formulations are subject to instabilities of different origins. The weak meshless formulation is suffering from the instability resulting from rank deficiency of the nodally integrated Galerkin meshless method, whereas, the global meshless methods in strong formulation have instability due to ill-conditioned system matrix. In a recent review paper of Liu [36], a detailed discussion can be found about weak and strong meshless formulations along with their associated advantages and disadvantages. Issues related to instability to the RPICM and some regularization techniques has been reported in [37–40,36].

Despite of difficulty in handling Neumann boundary conditions and shape parameter selection, recent advancements in various forms of localized meshless methods and their successful applications in thermo-fluid [41], macrosegregation with mesosegregates in binary metallic casts [42], freezing with natural convection [43], and wound healing modelling [44] have raised expectations of many researchers working in this area. Mathematical analysis of both forms of meshless methods and resolution of the above mentioned challenges are still in the early stages and deserve further focused efforts for bridging theoretical gapes necessary for the full pledged development of this new class of computational methods. One such comprehensive analysis has recently been reported in [45].

The current work is an initial step to highlight the above mentioned challenges faced by both strong and weak meshless formulation. In this work we employ a multi-resolution integration techniques [1,2] and explore their new applications in the context of numerical integration in the EFG method. Another contribution of the current work is a comparative performance of the local meshless strong formulation with meshless weak formulation for the elliptic boundary-value problems, for calculating displacement and strain. The localized RBFs meshless collocation method has been applied to problems with local features [46,47], such as problems with heterogeneity or cracks, dispersion of contaminants in an open channel flow [48,49] and wound healing modelling [44]. The LRBFCM is also used in variety of other applications such as multi-scale solidification modelling, numerical solution of convection-diffusion PDEs and other applications [35,42,50–55]. The problems considered in this paper are challenging for the weak meshless formulation due to some critical issues such as jump phenomena and ill-conditioned system matrices.

The organization of the paper is as follows. In Section 2, the EFG method formulation for one- and two- dimensional EBVPs is briefly described. In Section 3, the moving least squares (MLS) approximation is presented. In Section 4, quadrature rules based on Haar wavelets in one- and two- dimensions are described. In Section 5, the LRBFCM is presented briefly. In Section 6, several numerical examples of different types are discussed. Finally some conclusions of the paper are drawn in Section 7.

2. EFG method

In this section, the weak formulation based on EFG method for one- and two-dimensional EBVPs is briefly introduced.

2.1. One dimensional EFG formulation

Consider the following second order one-dimensional EBVP

$$-\frac{d}{dx}\left(\alpha\frac{du}{dx}\right)+\beta u+g(x)=0 \quad \text{in} \quad \Omega=(a,b), \tag{1}$$

with the boundary conditions:

$$\begin{aligned} u_{,x}n &= \bar{t} \quad \text{on} \quad \Gamma_t,\\ u &= \bar{u} \quad \text{on} \quad \Gamma_u, \end{aligned} \tag{2}$$

where $\alpha, \beta, g(x), \bar{t}$, and $\bar{u}$ are the known quantities of the problem and n is the unit outward normal on the boundary.

The weak formulation of (1) can be written as,

$$\int_a^b v(x)\left[-\frac{d}{dx}\left(\alpha\frac{du(x)}{dx}\right)+\beta u(x)+g(x)\right]dx=0, \tag{3}$$

where $v(x)$ is the weight function also called test function.

Integration by parts of Eq. (3) and rearrangement of the terms leads to the following weak formulation,

$$\int_a^b\left[\alpha\frac{v(x)}{dx}\frac{u(x)}{dx}dx+\alpha\frac{u(a)}{dx}v(a)\right]dx=\alpha\frac{u(b)}{dx}v(b) - \int_a^b \beta v(x)u(x) - \int_a^b v(x)g(x)dx. \tag{4}$$

One of the interesting property of weak formulation is the direct imposition of Neumann (natural) boundary conditions, whereas the Dirichlet (essential) boundary conditions is enforced most commonly by either using Lagrange's multiplier or penalty methods. Both the methods give same accuracy as they operate on the same variational form [56].

Similar to the Galerkin procedure, the trial function of the unknown solution $u(x)$ and the test function is chosen from the same space. In the EFG method, usually the moving least squares (MLS) approximation is chosen as a trial function with a vanishing weight function v over the essential boundary. Thus, eliminating the corresponding terms in Eq. (4) we have,

$$\int_a^b\left[\alpha\frac{v(x)}{dx}\frac{u(x)}{dx}dx+\beta v(x)u(x)\right]dx=\alpha\frac{u(b)}{dx}v(b) - \int_a^b v(x)g(x)dx. \tag{5}$$

Typically the weight function $v(x)$ used in Eq. (3) and in the MLS approximation is the same, which is usually chosen either cubic, quartic splines or Gaussian function. In this paper, we use the following cubic spline weight function:

$$v(x-x_I)=\begin{cases}\frac{2}{3}-4r^2+4r^3 & \text{for} \quad r\le\frac{1}{2},\\ \frac{4}{3}-4r+4r^2-\frac{4}{3}r^3 & \text{for} \quad \frac{1}{2}<r\le 1,\\ 0 & \text{for} \quad r>1,\end{cases} \tag{6}$$