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Well-balanced discontinuous Galerkin methods with hydrostatic reconstruction for the Euler equations with gravitation

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ABSTRACT

Many interesting astrophysical and atmospheric problems involve flows near the hydrostatic equilibrium state where the pressure gradient is balanced by the gravitational force. In this paper, we design high order well-balanced discontinuous Galerkin methods for the Euler equations with gravitation, which can preserve the discrete polytropic and isothermal hydrostatic balance states exactly. To achieve the well-balancedness, we propose to combine the numerical fluxes based on a generalized hydrostatic reconstruction, with an equilibrium state recovery technique and a novel source term approximation. Extensive one- and two-dimensional numerical examples are shown to demonstrate the performance of our well-balanced methods, and comparison with non-well-balanced results is included to illustrate the importance of maintaining the balance between pressure gradient and gravitational force numerically.

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1. Introduction

In this paper, we design high order well-balanced discontinuous Galerkin (DG) methods for the solutions of the Euler equations with gravitation

$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbf{I}_d) = -\rho\nabla\phi,\\
&E_t + \nabla\cdot((E+p)\mathbf{u}) = -\rho\mathbf{u}\cdot\nabla\phi,
\end{aligned} \tag{1.1}$$

that preserve their steady state solutions exactly in the discrete sense. Here $\mathbf{x}\in\mathcal{R}^d$ ($d=1,2,3$) is the spatial variable, ρ, $\mathbf{u}$, p denote the fluid density, the velocity, and the pressure, respectively. $E=\frac{1}{2}\rho\|\mathbf{u}\|^2+\rho e$ (e is internal energy) is the non-gravitational energy which includes the kinetic and internal energy of the fluid. The operators ∇, $\nabla\cdot$ and $\otimes$ are the gradient, divergence and tensor product in $\mathcal{R}^d$, respectively, and $\mathbf{I}_d$ denotes the identity matrix. The source terms on the right hand side of the equations represent the effect of the gravitational force, and $\phi=\phi(\mathbf{x})$ is the time independent

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gravitational potential. To close this system, the pressure p is linked to the density ρ and the interval energy e through an equation of state, denoted by $p = p(\rho, e)$. For example, the ideal gas law takes the form of

$$p = (\gamma - 1)\rho e = (\gamma - 1)\left(E - \rho\|\mathbf{u}\|^2/2\right), \tag{1.2}$$

where γ is the ratio of specific heats. This ideal gas law is used in the numerical examples section, but the methods presented in this paper are applicable beyond the ideal gas equation of state.

Euler equations under gravitational fields play an important role in modeling many interesting astrophysical and atmospheric phenomena, with examples including the simulation of supernova explosions, climate modeling and numerical weather forecasting. In these applications, we often encounter nearly steady state flows, which are small perturbation of the hydrostatic equilibrium states arising from the balance of the flux term and gravitational source term in (1.1). Two well-known hydrostatic equilibriums states of the Euler equations under gravitation are the isothermal and the polytropic equilibria, which will be explained in detail in Section 2. One computational challenge in simulating these nearly steady flows comes from the imbalance of numerical approximations to these terms, which will lead to truncation error that may be comparable with the size of the physical perturbation. As a result, the numerical solution may either oscillate around the equilibrium or deviate from the correct approximation. To resolve this problem, one may need to use an extremely refined mesh, which increases the computational cost and can become quite burdensome in multi-dimensional simulations. Well-balanced methods are designed to preserve these steady states solutions exactly up to the machine accuracy, and can effectively capture these small perturbations well even on relatively coarse meshes.

Study of well-balanced methods has attracted many attention in the past decade, and many well-balanced methods have been designed in the literature. Most of them are proposed for the shallow water equations over a non-flat bottom topology, which is another prototypical example of hyperbolic conservation laws with source term. We refer the readers to [2,12,1,15,21,31,26,30,29] and the references therein for some limited references in this context. Recently, some of these approaches have been extended to design well-balanced numerical methods for the Euler equations with gravitation. An early work can be found in [16], where the quasi-steady wave-propagation methods are applied to the Euler equations. Later, finite volume well-balanced methods have been proposed in [3] for the nearly hydrostatic flows in the numerical weather prediction. Gas-kinetic schemes have been extended to the multidimensional gas dynamic equations in [24,32,19], and well-balanced numerical methods were developed. Finite volume well-balanced schemes for the general hydrostatic equilibrium without any assumption of a thermal equilibrium are recently studied in [13,14]. Other related work can be found in [33,10,6]. The first high order version of well-balanced methods for the isothermal equilibrium of the Euler equations with gravitation is introduced in [28], based on a reformulation of the source term and a slightly modified weighted essentially non-oscillatory (WENO) reconstruction operator. The well-balanced approach based on reformulating the source term has been extended to DG methods in [17], to the nodal DG methods in [7], to the compact-reconstruction WENO methods for atmospheric flows in [11], and to the finite volume WENO methods in [18].

Another popular approach in designing well-balanced methods for the shallow-water equations is the hydrostatic reconstruction idea, first proposed in [1] and later appearing in many well-balanced methods including some high order ones. Numerical flux based on hydrostatic reconstruction, combined with novel well-balanced source term approximation, is an important idea in designing well-balanced DG methods for the lake at rest steady state [27,30], and for the general moving equilibrium state of the shallow water equations [4,5,25]. In this paper, we plan to extend the hydrostatic reconstruction idea to investigate novel well-balanced DG methods for the polytropic equilibrium of the Euler equations with gravitation, which appears in most of these astrophysical applications. Their extension to the isothermal equilibrium state will also be described. Our well-balanced DG methods are build upon the first order methods in [13]. In [13], second order extension has also been presented, and our methods can be viewed as their extension to arbitrary high order methods in the DG setting. To achieve the well-balancedness, we proposed to combine the numerical fluxes based on hydrostatic reconstruction, with the equilibrium state recovery technique and a novel source term approximation. The proposed DG methods can also be viewed as a generalization of the methods designed for balancing the shallow water equations with moving water equilibrium in [25], and are very different from the existing two well-balanced DG methods in [17,7].

This paper is organized as follows. In Section 2, we present the one dimensional model and its steady state solutions. In Sections 3 and 4, our well-balanced DG methods for the polytropic hydrostatic steady states of the Euler equations under gravitational filed are presented. We start with one dimensional problem, and then extend the proposed method to multi-dimensional case. Section 5 contains extensive numerical simulation results to demonstrate the behavior of our well-balanced DG methods for one- and two-dimensional Euler equations under gravitational field, verifying high order accuracy, the well-balanced property, and good resolution for smooth and discontinuous solutions. Some conclusions are given in Section 6.

2. Mathematical model

In one spatial dimension, the Euler equations (1.1) reduce to the form of

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&(\rho u)_t + (\rho u^2 + p)_x = -\rho\phi_x,\\
&E_t + ((E + p)u)_x = -\rho u\phi_x,
\end{aligned} \tag{2.1}$$

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