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High-order upwind schemes for the wave equation on overlapping grids: Maxwell's equations in second-order form

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ABSTRACT

High-order accurate upwind approximations for the wave equation in second-order form on overlapping grids are developed. Although upwind schemes are well established for first-order hyperbolic systems, it was only recently shown by Banks and Henshaw [1] how upwinding could be incorporated into the second-order form of the wave equation. This new upwind approach is extended here to solve the time-domain Maxwell's equations in second-order form; schemes of arbitrary order of accuracy are formulated for general curvilinear grids. Taylor time-stepping is used to develop single-step space-time schemes, and the upwind dissipation is incorporated by embedding the exact solution of a local Riemann problem into the discretization. Second-order and fourth-order accurate schemes are implemented for problems in two and three space dimensions, and overlapping grids are used to treat complex geometry and problems with multiple materials. Stability analysis of the upwind-scheme on overlapping grids is performed using normal mode theory. The stability analysis and computations confirm that the upwind scheme remains stable on overlapping grids, including the difficult case of thin boundary grids when the traditional non-dissipative scheme becomes unstable. The accuracy properties of the scheme are carefully evaluated on a series of classical scattering problems for both perfect conductors and dielectric materials in two and three space dimensions. The upwind scheme is shown to be robust and provide high-order accuracy.

1. Introduction

Partial differential equations governing wave propagation problems are typically formulated in either first-order form as a first-order hyperbolic system, or in second-order form (e.g. the classical wave equation $u_{tt} = c^2 \Delta u$). As discussed in [1], treating the second-order form directly can often have a number of advantages. For example, there may be fewer dependent variables (e.g. going from six to three dependent variables for Maxwell's equations) and in some cases fewer constraint equations (e.g. the Saint-Venant compatibility conditions for linear elasticity written in first-order form). An additional advantage of the second-order form is that it is straightforward to develop compact, high-order accurate, and

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discretely self-adjoint approximations to the Laplace operator [2]. In contrast, the first-order form often requires the use of staggered grids for finite difference methods [3], or special edge elements for finite-element methods [4] to avoid nontrivial null-spaces, which can lead to numerical difficulties in the form of undamped or growing highly oscillatory modes.

Until recently, one significant advantage of using the first-order system form of the equations has been the availability of robust and accurate numerical schemes based on upwinding, or more generally schemes which incorporate the characteristic structure of the PDE into the discrete operators. The first instance of such a scheme is the one developed by Courant, Isaacson, and Rees [5], which was built around the idea of characteristics but did not directly invoke the idea of "upwind". It was the landmark 1959 paper of Godunov [6] which spoke of directly incorporating upwinding by embedding the exact solution of the Riemann problem into the numerical technique. The core idea in this approach is that the solution to the Riemann problem correctly accounts for the directional transport, the so-called wind, of characteristic quantities. Based on these fundamental ideas there have been many powerful extensions including for example, the flux-corrected transport method [7], semi-Lagrangian methods [8–10], the piecewise-parabolic-method (PPM) [11], essentially-non-oscillatory (ENO) schemes [12,13], discontinuous Galerkin (DG) methods [14], and weighted-essentially-non-oscillatory (WENO) methods [15]. In the context of the present work, it is also important to note the practical utility of using upwinding for the first-order formulation of Maxwell's equations, e.g. [16–18]. In fact, as discussed in [19], there exists a direct relationship between upwind numerical methods and schemes that directly incorporate an artificial dissipation; such artificial dissipation schemes are still in wide-spread use and trace their origins to the artificial viscosity scheme of Richtmeyer and von Neumann [20–22]. However, there are significant advantages of directly using ideas of upwinding, as opposed to ad-hoc addition of an artificial dissipation, in that the form and scaling of the dissipation in upwind schemes is a natural by-product of the method and as a result it is not necessary to tune artificial parameters or the form of the dissipation operators.

Despite their success for the first-order system, upwind methods directly applicable to wave equations posed in second-order form were not discussed until recently in [1]. The key idea introduced in [1], which was essentially similar to the original approach of Godunov [6], was based on embedding the well-known d'Alembert solution into the discretization. This d'Alembert solution plays a similar role to the Riemann problem for the first-order system and provides a mechanism to account for the wave nature of the solution. Following the established procedure for upwind treatments for the first-order form, a localized expression of the upwind flux was derived that enables easy application to a wide class of problems including those in multiple space dimensions and those with variable coefficients. In [1], schemes with order-of-accuracy ranging from one to six were developed and the numerical approximations were found to be quite well behaved even for very difficult problems involving weak solutions with jumps; a particularly difficult case for the second-order form of the equations. Subsequently in [23], a preliminary extension of the scheme to overlapping grids was performed. Similar to the experience with upwind methods for the first-order system, the second-order system upwind methods were found to be stabilizing even for difficult cases involving overlapping grids where dissipation-free schemes for wave equations are known to exhibit instabilities [24]. Recently, a similar construction was followed to develop discontinuous Galerkin methods for the second-order form of the equations in [25].

Maxwell's equations in the time-domain are usually solved in first-order form and there are a wide class of methods that have been developed for their solution, including those based on finite difference, spectral, pseudo-spectral, finite-element, and discontinuous Galerkin, amongst others. The literature is very broad; for a good introduction see, for example, the review by Hestaven [26] or the references in the books by Taflove [27] and Cohen [28]. Note that overlapping grids have previously been used for the solution of Maxwell's equations in first order form by Driscoll and Fornberg [29] using a hybrid pseudo-spectral finite-difference scheme. In related work, the first-order formulation was also used in early mesh refinement strategies [30], and more recent adaptive mesh refinement [31]. However, it was later found that reflection from AMR boundaries was kept at more manageable levels by using upwind discretizations of the first-order formulation, e.g. [32–34]. Despite the many excellent schemes that have been developed, there still remains a need in some applications for more efficient schemes on complex geometries that are both robust and high-order accurate. Toward this end, a scheme based on upwind schemes (for robustness) and overlapping grids (for efficiency) may prove to be useful. Therefore, in the current work we extend the ideas for high-order upwind methods outlined in [1,23] to discretize Maxwell's equations in second-order form on overlapping grids. In this framework, geometric complexity introduced by both physical boundaries (e.g. perfectly conducting boundaries) and interfaces between dielectric materials will be treated using composite overlapping grids. A simple motivating example, which involves the interaction of an impinging electromagnetic wave on one perfectly conducting disk and one dielectric disk, is illustrated in Fig. 1. As in [2], the governing equations are written as a system of scalar wave equations for the components of the electric field which are coupled at domain boundaries and material interfaces. In contrast to [2] which relies on a simple artificial dissipation, we achieve stabilization against overlapping grid instabilities through the use of upwind discretizations. As will be demonstrated, these upwind discretizations are stable for all overlapping grid configurations investigated without any adjustable artificial viscosity. This is true even for the difficult case where the boundary fitted grids use a fixed number of grid points as the composite grid is refined. For such a refinement process the overlapping grid interpolation boundary approaches the physical boundary with the result that overlapping grid instabilities become more pronounced [24]. This situation is theoretically investigated here in one dimension using the normal mode stability theory of Gustafsson, Kreiss, and Sundström [35,36], and the upwind dissipation is found to be stabilizing. Indeed the upwind discretizations are evidently stable and accurate in higher dimensions even for refinements that yield thin boundary grids. On the other hand the artificial dissipation approach in [2] requires an increasingly large artificial viscosity parameter which results in a reduction in observed accuracy.