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Uniform dispersion reduction schemes for the one dimensional wave equation in isotropic media

Yajun An [1]

Pacific Lutheran University, Department of Mathematics, Tacoma, USA

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ABSTRACT

Finite difference (FD) methods for the wave equation in general suffer from numerical dispersion. Although FD methods based on accuracy give good dispersion at low frequencies, waves tend to disperse for higher wavenumber. In this work, we will give a unified methodology to derive dispersion reduction FD schemes for the one dimensional wave equation, and this new method can reduce dispersion error uniformly for all wavenumbers up to the Nyquist. Stability criteria are given, and stability analysis is done for each generated scheme.

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Introduction

Finite Difference (FD) schemes have been used widely in computing approximations for partial differential equations for wave propagation, as they are simple, flexible and robust [8]. The method has thus become very popular, and one example is the second-order accurate center difference Leap-frog scheme. However, even for stable and accurate schemes, waves in the numerical schemes can propagate at different wave speeds than in the true medium, namely, there exists a numerical dispersion error [1]. Traditionally, FD schemes are designed by forcing accuracy conditions, and in spite of the advantages mentioned above, such schemes suffer from numerical dispersion errors. These errors can result in poor approximation to the wave speed for high wavenumbers, and thus cause poor approximations to the wave propagation as time spans get larger.

Many methods have been used for the purpose of reducing dispersion error: increasing the sampling rate and using higher order accuracy [14]. Staggered grids have also been introduced in [20,21]. Liu [17] developed schemes that give optimized least-squares based spatial FD coefficients that incorporate accuracy and exact dispersion requirements. Finkelstein and Kastner in [4] and [5] proposed a unified methodology for deriving new schemes that can accommodate arbitrary requirements for reduced phase or group velocity dispersion errors, defined over any region in the frequency domain. Such schemes are based on enforcing exact phase or group velocity at certain preset wavenumbers. This method has been shown to reduce dispersion errors at large wavenumbers. Liu and Sen [16] gave explicit expression of such schemes in their work. Kosloff et al. proposed the construction of second-derivative operators by the Remez algorithm [18] in [11,12].

In this paper, we study the construction and behaviors of FD schemes designed to reduce numerical dispersion error for all wavenumbers based on [4,5,16,12]. We give a theorem [2] that proves the existence of schemes proposed by [4,5,16]. Furthermore, from the existence of such schemes, we can show that schemes which reduce the dispersion error uniformly

E-mail address: anya@plu.edu.

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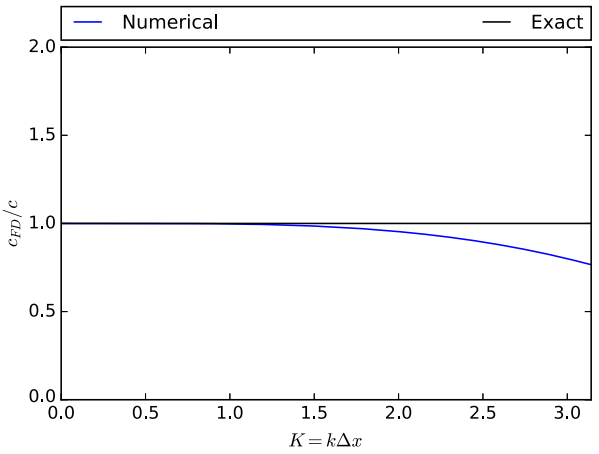
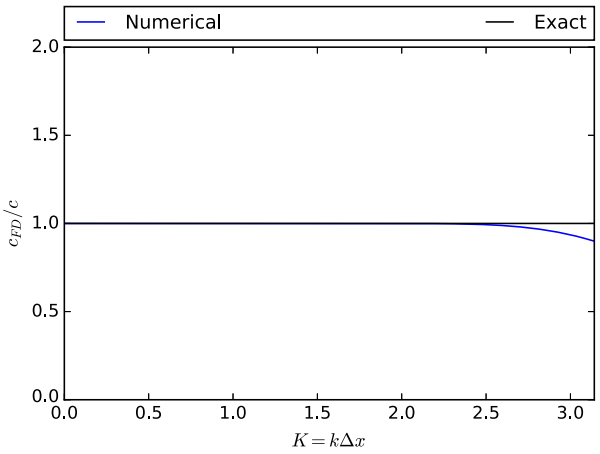


(a) $M = 2$, max error=0.23390380741442984



(b) $M = 12$, max error=0.10042897825001129

Fig. 1. Comparison of dispersion curve for different stencil widths M, $\gamma = 0.6$. Wider stencil width gives smaller dispersion error. Schemes obtained from Eq. (7).

in an interval of the frequency domain can be constructed from a Remez-like algorithm [7,9,10]. In these new schemes we propose, we can also specify wavenumbers where exact phase or group dispersion relation can be satisfied [2]. For an incoming signal consisting of waves of different wavenumbers, our scheme gives better synthetic wave propagation. We apply our methodology to 1D homogeneous, and lossless wave equations in this work.

1. Order of accuracy and spectral order of accuracy

In this paper, we study the one-dimensional wave equation

$$(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})u(x,t) = 0 \tag{1}$$

in a homogeneous, lossless, boundless and source free medium. Here c is the constant wave speed (we take $c = 1$). The traditional way is to use a (3,3) leap-frog scheme to approximate the above equation:

$$U_j^{n+1} - 2U_j^n + U_j^{n-1} - \gamma^2\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right) \approx 0. \tag{2}$$

Here $\gamma = c\dfrac{\Delta t}{\Delta x}$ denotes the CFL number [3], U_{j+m}^{n+l} denotes the grid function approximating $u(x + m\Delta x, t + l\Delta t)$ and U_j^n the center point at the present time $u(x,t)$. This discretization enforces 2nd order accuracy at (x,t). Let k denote the physical wave number and ω the physical frequency and $\Omega = \omega\Delta t$ and $K = k\Delta x$ denote the normalized numerical frequency and normalized numerical wavenumber satisfying Eq. (3). Here $K \in [0, \pi]$, with $K_{max} = \pi$ being the Nyquist wavenumber. Applying the discrete Fourier transform ($x \to K$, $t \to \Omega$) gives the dispersion relation

$$1 - \cos(\Omega) - \gamma^2 + \gamma^2\cos(K) \approx 0. \tag{3}$$

We solve for the numerical phase velocity of the wave here, obtaining

$$c(K) = \frac{c}{\gamma K}\arccos(1 - \gamma^2 + \gamma^2\cos(K)), \tag{4}$$

whereas the true phase velocity is $c = \dfrac{\omega}{k}$ (we only consider right propagating wave). Energy propagates at the group velocity, computed by $c_g = \dfrac{d\omega}{dk}$ [19].

Eq. (2) can be generalized to a $(3, 2M+1)$ stencil, using $2M+1$ points in space. As shown in Fig. 1, increasing the number of points in space increases the accuracy of phase velocity for higher wavenumbers, and the generalized finite difference equation is:

$$U_j^{n+1} + U_j^{n-1} + \sum_{m=0}^{M} c_m(U_{j+m}^n + U_{j-m}^n) = 0, \tag{5}$$

where $\{c_m\}_{m=0}^M$ are undetermined coefficients.

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