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Adjoint-based sensitivity analysis of low-order thermoacoustic networks using a wave-based approach

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ABSTRACT

Strict pollutant emission regulations are pushing gas turbine manufacturers to develop devices that operate in lean conditions, with the downside that combustion instabilities are more likely to occur. Methods to predict and control unstable modes inside combustion chambers have been developed in the last decades but, in some cases, they are computationally expensive. Sensitivity analysis aided by adjoint methods provides valuable sensitivity information at a low computational cost. This paper introduces adjoint methods and their application in wave-based low order network models, which are used as industrial tools, to predict and control thermoacoustic oscillations. Two thermoacoustic models of interest are analyzed. First, in the zero Mach number limit, a nonlinear eigenvalue problem is derived, and continuous and discrete adjoint methods are used to obtain the sensitivities of the system to small modifications. Sensitivities to base-state modification and feedback devices are presented. Second, a more general case with non-zero Mach number, a moving flame front and choked outlet, is presented. The influence of the entropy waves on the computed sensitivities is shown.

1. Introduction

The F-1 engines used in the Saturn V rockets were the subject of an expensive, but ultimately successful, attempt to mitigate combustion oscillations. More than 3200 full-scale tests were required [1]. Today, this cost would be prohibitive, which demonstrates the need for robust analytical tools to predict the onset of thermoacoustic oscillations and methods to control them.

Network models using wave-based approaches have been widely used in thermoacoustics [2–4]. As described by Dowling and Stow [5] a thermoacoustic network is a collection of acoustic elements such as ducts, plenums, combustors, boundaries, and a combustion zone, which is normally assumed to be compact. The elements' mean flow quantities are often considered to be homogeneous in each network element. Both mean flow quantities and fluctuations are related across elements by jump relations for the mass, momentum, and energy.

The primary objective of combining adjoint methods with stability analysis is to calculate the eigenvalues and their sensitivities to small modifications to the system, which can be caused by a variation of a parameter or the introduction of a feedback device (see e.g. [6,7]).

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In thermoacoustics, nonlinear adjoint looping was used by Juniper [8] to find the smallest initial perturbation that could cause triggering of self-sustained oscillations in an electrically heated Rijke tube. The first application of adjoints in eigenvalue sensitivity analysis was performed by Magri and Juniper [9], who modelled a time-delayed thermoacoustic system in low Mach number conditions. Using Galerkin methods, they studied the eigenvalue sensitivity to (i) any of the parameters of the system (base state sensitivity) and (ii) generic passive control devices (feedback sensitivity, also known as structural sensitivity). One outcome was finding that a fine mesh in the second half of the tube would help to stabilize the system. This was tested experimentally by Rigas et al. [10], who measured the growth rate and the frequency shift in the presence of the passive control device. The growth rate shift was predicted accurately by adjoint analysis applied to the model. There was, however, some discrepancy in the frequency shift, which was due to limitations of the thermoacoustic model, rather than the sensitivity analysis.

Wave-based methods produce a nonlinear eigenvalue problem of the form [11–13]:

$$\mathrm{L}(s, \mathbf{p})\boldsymbol{q} = \mathbf{0}, \tag{1}$$

where s is the eigenvalue and $\mathbf{p}$ are the parameters of the system such as the reflection coefficients, time delays and heat source parameters. The adjoint function can be defined by means of a bilinear form $[\cdot,\cdot]$ such that for arbitrary $\boldsymbol{a}, \boldsymbol{b}$:

$$[\boldsymbol{a}, \mathrm{L}\boldsymbol{b}] - \left[\mathrm{L}^{+}\boldsymbol{a}, \boldsymbol{b}\right] = \text{constant}, \tag{2}$$

where L^{+} is the adjoint operator. An operator L is said to be normal if its eigenfunctions $\boldsymbol{q}$ are orthogonal, or equivalently, if $\mathrm{L}\mathrm{L}^{+} = \mathrm{L}^{+}\mathrm{L}$. Clearly, an operator is normal if it is self-adjoint, i.e. $\mathrm{L}^{+} = \mathrm{L}$. The equations governing duct acoustics, without considering boundary conditions, obey the wave equation and are self-adjoint (see e.g. [14]). Nicoud et al. [15] demonstrated that thermoacoustic eigenfunctions are not orthogonal to each other, meaning that thermoacoustic systems are non-normal. Wieczorek et al. [16] showed that non-normal effects in thermoacoustics increase with the mean flow velocity. Therefore, with a mean flow, thermoacoustic systems are expected to be even less normal.

Depending on the sensitivity information desired, the operator, L, needs to be perturbed. Two different types of perturbation are defined:

- when the parameters $\mathbf{p}$ are perturbed, the resulting sensitivity is named *base state sensitivity*;
- when the system is perturbed by adding a small feedback mechanism, which is linearly proportional to one of the state variables of vector $\boldsymbol{q}$, the resulting sensitivity is called *feedback sensitivity* (also known as *structural sensitivity* in Giannetti and Luchini [17] and Magri and Juniper [9]). Feedback mechanisms that cause mass addition, momentum addition, and/or energy addition are considered.

In this paper we extend adjoint-based sensitivity analysis to wave-based thermoacoustic models, which produce a nonlinear eigenvalue problem. Throughout this study, we focus on first-order perturbations. Higher-order perturbation studies have been performed by Magri [12], Magri et al. [13,18], Mensah and Moeck [19], Silva et al. [20] but are not considered further here. First, we consider a zero-Mach number thermoacoustic system to show the symmetries between the direct and adjoint eigenfunctions, which are harder to see when the mean flow and entropy waves are included. The sensitivities are calculated using both continuous and discrete adjoint approaches, and the computational/physical advantages and disadvantages of these two methods are discussed. In the second part of this paper, the methods are extended to include a mean flow, a moving flame front, and a choked outlet in a more realistic combustor model. The paper ends with a concluding discussion.

2. Thermoacoustic model with zero mean flow

A one dimensional network model composed of a duct of length L_n with a compact heat source located at $x = b$ is considered. The model assumes homogeneous properties along each segment, hence the heat source splits the domain into two segments as shown in Fig. 1. Each segment is governed by a similar set of equations, which are connected by the jump conditions established by the heat source.

2.1. Governing equations

The governing equations for the ducts of the thermoacoustic system are given by the continuity, momentum, and the energy equations, neglecting viscosity and heat conduction:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{3a}$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \tag{3b}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x} = 0. \tag{3c}$$