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Multi-fidelity stochastic collocation method for computation of statistical moments ☆

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ABSTRACT

We present an efficient numerical algorithm to approximate the statistical moments of stochastic problems, in the presence of models with different fidelities. The method extends the multi-fidelity approximation method developed in [18,26]. By combining the efficiency of low-fidelity models and the accuracy of high-fidelity models, our method exhibits fast convergence with a limited number of high-fidelity simulations. We establish an error bound of the method and present several numerical examples to demonstrate the efficiency and applicability of the multi-fidelity algorithm.

1. Introduction

In recent years, many efforts have been devoted to the development of efficient numerical methods for uncertainty quantification (UQ). In practical computations, the most widely used method is stochastic collocation, as it is nonintrusive sampling based and allows one to use existing deterministic codes. Unfortunately, this number of deterministic simulations required by accurate stochastic collocation methods grows very rapidly for high dimensional random inputs — the curse of dimensionality. For large scale simulations, the computational cost can become prohibitive, as the computation of each individual deterministic sample is highly costly. Many options have been investigated to tackle this challenge. For example, methods that explore more efficient sampling strategies using sparse grids, adaptivity, smoothness or sparsity of the solutions, cf. [25,2,1,11,5,8,9,12,16,17,19,20,23]. There is also a recent surge of interest in multilevel Monte Carlo method, which uses the hierarchy models by physical space refinement to achieve variance reduction in random space, cf., [14,4,3,7,21]. Other approaches to achieve variance reduction have also been presented, cf., [6,22].

In this paper we focus primarily on the computation of solution statistics using models with different fidelities. In particular, we focus on the case with one high-fidelity model and one low-fidelity model. Here, the high-fidelity model is able to produce high resolution solution to the underlying physical problem. The simulation cost is high, thus preventing us from using the standard sampling strategy (Monte Carlo, sparse grids, etc.). The low-fidelity model, on the other hand, is not highly accurate but can capture the essential behavior of the underlying problem. It is computationally cheap and can be sampled a large number of times. Typically, the low-fidelity models are constructed using simplified physics and/or

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much coarser discretization. Examples are abundant in many problems, for example, the fine-scale versus the coarse grained models in multi-scale problems.

We present an efficient stochastic collocation algorithm for computing solution statistics using a high-fidelity model and a low-fidelity model. A distinct feature of our method is that it "seperate" the low-fidelity solutions and high-fidelity solutions. It uses the low-fidelity solutions, which consist of a large number of samples, to construct a best approximation of the target solution statistics (mean, variance, etc.), and then apply the best approximation to the high-fidelity samples. Our method is essentially a "learning" algorithm, where the low-fidelity samples are used to "train" the best approximation. It is different from most of the existing methods, which usually achieve improved performance via variance reduction or by exploring the hierarchical structure (if available) of the models. The current method is an extension of the method developed in [18,26], where the same training idea was first proposed and used to predict the solutions at arbitrary sample locations. A straightforward way to use the technique of [18,26] to compute solution statistics is to compute the bi-fidelity solutions at every sample points and then compute the statistics. Although computing each bi-fidelity solution is efficient, computing such solutions at a large number of samples becomes expensive. The major contribution of this paper is to present a mathematically equivalent algorithm that directly computes the solution statistics and bypasses the step of a large number of bi-fidelity computations. Here we show that the method can be highly efficient in approximating the statistics (mean, variance, etc.) of the underlying stochastic problem. We establish an error bound of the method and use extensive numerical examples to demonstrate its performance. In the examples with varying multiple dimensions, accuracy solutions can be obtained by $O(10)$ number of high-fidelity simulation samples.

2. Problem setup

Let w be the solution of a system of governing equations in a bounded spatial domain $D \subset \mathbb{R}^\ell$, $\ell = 1, 2, 3$, and a random parameter domain $I_Z \subseteq \mathbb{R}^d$, $d \geq 1$. For general discussion we do not assume any specific form of the governing equations. We are interested in a quantity-of-interest (QoI), which is a function of the solution w, i.e.,

$$v = q(w) : \bar{D} \times I_Z \to \mathbb{R}. \tag{2.1}$$

Hereafter we denote $x = (x_1, \ldots, x_\ell)$ the spatial variable and $z = (z_1, \ldots, z_d)$ the random variable. Let $\rho : I_z \to \mathbb{R}^+$ be the probability distribution function of z. We are interested in evaluating the statistical average of the QoI, $v : \bar{D} \to \mathbb{R}$,

$$v(\cdot) = \mathbb{E}[v] = \int v(\cdot, z)\rho(z)dz. \tag{2.2}$$

For example, when $v = w^k$, $k \geq 1$, it stands for the k-th moment of the solution.

2.1. Numerical approximations

For numerical approximation, we seek an approximate solution u in a linear subspace V for any fixed random variables,

$$u : I_z \to V. \tag{2.3}$$

Obviously, the choice of the linear subspace V depends on the chosen numerical method. We assume that the numerical method is deterministic and satisfies

$$u(\cdot, z) \approx v(\cdot, z), \qquad \forall z \in I_Z,$$

in a proper norm in the physical space.

Since the solution dependence in the random space can also be complex, the mean operator $\mathbb{E}$ in (2.2) also needs to be approximated. In this paper we focus on linear sampling based approximation, which is the predominant approach in practice. Let $\Theta = \{z_1, \ldots, z_m\} \subset I_Z$ be a set of samples, then for any integrable function $f : I_Z \to \mathbb{R}$ we define

$$\widetilde{\mathbb{E}}[f; \Theta] := \sum_{i=1}^{m} w_i f(z_i) \approx \mathbb{E}[f], \tag{2.4}$$

where w_i is the weight at the sample z_i, for $i = 1, \ldots, m$. For example, the standard Monte Carlo method has an uniform weight $w_i \equiv 1/m$, whereas for most cubature rules the weights are non-uniform. Hereafter we assume the weights satisfy

$$\sum_{i=1}^{m} w_i = 1, \qquad \|w\|_{\ell_2} < \infty, \tag{2.5}$$

where $\|w\|_{\ell_2}$ is the 2-norm of $w = (w_1, \ldots, w_m)$. Although it is highly desirable to have $w_i > 0$, this is not the case for many cubature rules.

With the approximations in both the physical space and the random space, we have

$$\mu(\cdot) = \widetilde{\mathbb{E}}[u; \Theta] \approx v(\cdot) \tag{2.6}$$

as an approximation to the true statistical average (2.2).