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Enhancement of flow measurements using fluid-dynamic constraints

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ABSTRACT

Novel experimental modalities acquire spatially resolved velocity measurements for steady state and transient flows which are of interest for engineering and biological applications. One of the drawbacks of such high resolution velocity data is their susceptibility to measurement errors. In this paper, we propose a novel filtering strategy that allows enhancement of the noisy measurements to obtain reconstruction of smooth divergence free velocity and corresponding pressure fields which together approximately comply to a prescribed flow model. The main step in our approach consists of the appropriate use of the velocity measurements in the design of a linearized flow model which can be shown to be well-posed and consistent with the true velocity and pressure fields up to measurement and modeling errors. The reconstruction procedure is then formulated as an optimal control problem for this linearized flow model. The resulting filter has analyzable smoothing and approximation properties. We briefly discuss the discretization of the approach by finite element methods and comment on the efficient solution by iterative methods. The capability of the proposed filter to significantly reduce data noise is demonstrated by numerical tests including the application to experimental data. In addition, we compare with other methods like smoothing and solenoidal filtering.

1. Introduction

Since numerous years the visualization of flow fields has had a significant impact on the systematic understanding and development of fluid-dynamic models as well as on the calibration and verification of computational methods. While traditional experimental techniques were able to provide only partial information about the flow field, novel measurement modalities such as particle tracking, tomographic particle imaging, or magnetic resonance velocimetry deliver spatially resolved three-dimensional velocity measurements [1–4]. In principle, these new methods therefore allow to image complex flow patterns in a wide range of engineering applications and even in biological in-vivo studies.

Distributed flow measurements provide valuable information about simple and complex flows, but they are typically contaminated by measurement errors which limit their usability in practice to some extent. In order to make the flow measurements more suitable for further analysis, e.g., for model discrimination or for the assessment of derived quantities like pressure drop or wall shear stress, some sort of data post-processing is required [5]. A widely used technique in this context is *data smoothing* which can be accomplished, for instance, by Tikhonov regularization [6,7] given by

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$$\min_{\mathbf{u}} \|\mathbf{u} - \mathbf{u}^\delta\|^2 + \alpha\|\nabla\mathbf{u}\|^2. \tag{1}$$

Here and below $\mathbf{u}^\delta$ denotes the flow measurements and the minimizer will be the enhanced velocity field. Note that the penalization of the gradient term leads to a smoothed velocity reconstruction. The choice of the regularization parameter α allows a certain trade-off between smoothness and fit to the data. The underlying quadratic minimization problem can be solved efficiently by Fourier transform or multigrid iterative solvers which makes this filter very efficient in practice. Note that the above procedure and also various other image processing methods [8] successfully reduce high frequency components in the noisy measurements but do not utilize any information about the underlying physics.

In many applications, the flow under consideration is incompressible and one might want to incorporate such prior knowledge into the reconstruction process. Requiring the improved velocity field to be divergence free and using a smoothing procedure similar to above, we obtain a constrained minimization problem

$$\min_{\mathbf{u}} \|\mathbf{u} - \mathbf{u}^\delta\|^2 + \alpha\|\nabla\mathbf{u}\|^2 \qquad \text{s.t.} \tag{2a}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2b}$$

This quadratic programming problem can again be solved efficiently by iterative methods. Various computational strategies leading to related divergence free reconstructions have been investigated recently in the literature under the name *divergence-free* or *solenoidal filtering*; see e.g. [9–14]. Let us note that, although some noise reduction has been observed even for the case $\alpha = 0$, the divergence constraint alone does formally not guarantee smoothness of the reconstruction. This can be seen from the Helmholtz decomposition of vector fields [15] and will be illustrated by numerical tests below.

A natural extension of the solenoidal filtering approach, which takes into account only the incompressibility, is to incorporate into the reconstruction process also a model for the momentum balance. Since distributed measurement techniques typically acquire time averaged data, it seems reasonable to assume steady flow conditions and to consider, as a first step, the stationary Navier–Stokes equations as the governing physical model. The reconstruction could then be defined via

$$\min_{\mathbf{f},\mathbf{u},p} \|\mathbf{u} - \mathbf{u}^\delta\|^2 + \alpha\|\mathbf{f}\|^2 \qquad \text{s.t.} \tag{3a}$$

$$-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0. \tag{3b}$$

Here and below $\nu > 0$ denotes the viscosity parameter which is assumed to be constant here. Additional boundary conditions have to be prescribed for a complete specification of the flow model. The residual $\mathbf{f}$ in the momentum equation measures the deviation from an idealized flow model due to unmodeled effects like time dependence or non-Newtonian behavior. Since a prescribed flow model is satisfied by the reconstructed fields, one may call such an approach a *fluid-dynamically consistent* filter. Due to the presence of the differential terms in the flow model (3b), the reconstruction will be smooth automatically and no additional penalization of the velocity gradients is required. Moreover, some information about the pressure p is obtained.

The system (3a)–(3b) has the form of an optimal control problem governed by the Navier–Stokes equations. Such problems have been investigated extensively in the literature; see e.g. [16–19] for steady and [20–24] for unsteady flow. Note that the nonlinearity in the momentum equation poses severe challenges, both, for the analysis and for the numerical solution. It is well-known, for instance, that the Navier–Stokes system admits a unique solution only for sufficiently small data [25,26]. Moreover, due to the nonlinear constraints, the optimization problem (3a)–(3b) is non-convex and may in general have many local minima. Both aspects make the computational solution demanding or even infeasible.

In this paper, we therefore propose a strategy that allows us to take advantage of the benefits and at the same time to overcome the drawbacks of the previous approach. The basic step here is to use the distributed velocity measurements $\mathbf{u}^\delta$ in order to replace the nonlinear term in the momentum equation by some linearization; one may think of $\mathbf{u}^\delta \cdot \nabla\mathbf{u}$ as an approximation for $\mathbf{u}\cdot\nabla\mathbf{u}$, although such a simple choice would not yield a well-posed problem in general due to lack of smoothness in the data. A proper linearization of the convective term $\mathbf{u}\cdot\nabla\mathbf{u}$ will however allow us to replace the nonlinear problem (3a)–(3b) by a quadratic optimal control problem with linearized constraints which has a unique minimizer that can be computed efficiently. The way in which we use the measurements in the governing equations is closely related to the *equation error method*, which is well-established in the context of parameter estimation [27,28].

In summary we thus obtain a well-posed and analyzable reconstruction method that produces a smooth divergence free velocity field and a corresponding pressure distribution which agree well with the measurements and at the same time approximately satisfy the prescribed fluid-dynamic model. A proper choice of the regularization parameter α will allow us to find a good balance between data fit and model errors.

The remainder of the manuscript is organized as follows: In Section 2, we introduce the linearized fluid flow model underlying our reconstruction approach and we formulate appropriate boundary conditions. We then establish the well-posedness of the linearized flow problem and derive error estimates for the linearization step. In Section 3, we introduce the optimal control problem which serves as mathematical formulation of our reconstruction procedure. We prove existence and uniqueness of minimizers, provide some error estimates, and highlight a direct connection to the solenoidal filtering. Our approach is formulated in infinite dimensions and some discretization strategy is required in order to obtain implementable algorithms. In Section 4, we therefore outline the discretization by finite element methods and we briefly discuss