Journal of Computational Physics 337 (2017) 379–402

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Revisiting the spectral analysis for high-order spectral discontinuous methods

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ARTICLE INFO

Article history:
Received 29 April 2016
Received in revised form 20 January 2017
Accepted 18 February 2017
Available online 27 February 2017

Keywords:
Space–time spectral analysis
Spectral discontinuous
Finite Difference
Aeroacoustics
Aliasing
Matrix Power Method

ABSTRACT

The spectral analysis is a basic tool to characterise the behaviour of any convection scheme. By nature, the solution projected onto the Fourier basis enables to estimate the dissipation and the dispersion associated with the spatial discretisation of the hyperbolic linear problem. In this paper, we wish to revisit such analysis, focusing attention on two key points. The first point concerns the effects of time integration on the spectral analysis. It is shown with standard high-order Finite Difference schemes dedicated to aeroacoustics that the time integration has an effect on the required number of points per wavelength. The situation depends on the choice of the coupled schemes (one for time integration, one for space derivative and one for the filter) and here, the compact scheme with its eighth-order filter seems to have a better spectral accuracy than the considered dispersion-relation preserving scheme with its associated filter, especially in terms of dissipation. Secondly, such a coupled space–time approach is applied to the new class of high-order spectral discontinuous approaches, focusing especially on the Spectral Difference method. A new way to address the specific spectral behaviour of the scheme is introduced first for wavenumbers in $[0, \pi]$, following the Matrix Power method. For wavenumbers above π, an aliasing phenomenon always occurs but it is possible to understand and to control the aliasing of the signal. It is shown that aliasing depends on the polynomial degree and on the number of time steps. A new way to define dissipation and dispersion is introduced and applied to wavenumbers larger than π. Since the new criteria recover the previous results for wavenumbers below π, the new proposed approach is an extension of all the previous ones dealing with dissipation and dispersion errors. At last, since the standard Finite Difference schemes can serve as reference solution for their capability in aeroacoustics, it is shown that the Spectral Difference method is as accurate as (or even more accurate) than the considered Finite Difference schemes.

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http://dx.doi.org/10.1016/j.jcp.2017.02.043
0021-9991/

1. Introduction

Because of the continuous growth of available computational resources during the last decade, there was an increased interest in performing Large Eddy Simulation – LES – to solve industrial problems. Among these problems, aeroacoustics requires to compute and to transport accurately pressure waves around complex geometries and over a long distance. Many classes of schemes were proposed to perform LES during the last 30 years, depending on the underlying mathematical framework considered to discretise the Navier–Stokes equations.

First, in the context of Finite Difference – FD – formalism, high-order centred schemes for structured grids were built following the Taylor's expansion technique and their accuracy was compared with the one of spectral methods [1]. Any (high) order of accuracy can be attained but the number of degrees of freedom to update the solution at one point can be large. Two optimisations of FD schemes were introduced: the compact formulation of Lele [2] that leads to a spectral-like resolution and the Dispersion–Relation–Preserving – DRP – technique of Tam and Webb [3] dedicated to aeroacoustics. The compact approach links several derivatives with unknowns located closely. By this way, a linear (implicit) system of equations links all derivatives with unknowns. For a given accuracy, the stencil of DRP schemes is larger than for the standard FD approach and the extra unknowns enable to control the numerical properties of the scheme: dissipation and dispersion. Both approaches being centred, they are non dissipative and a filter stabilizes the computations by dissipating wavenumbers. In this paper, we consider compact and DRP schemes as standard ingredients for LES and it is assumed that they will provide reference results.

More recently, a new generation of high-order techniques denoted as *spectral discontinuous* emerged. Following the pioneering work of Reed and Hill [4], the Discontinuous Galerkin – DG – formulation was first applied to hyperbolic equations by Cockburn, Shu and co-authors [5–7] and opened many years of research and papers (see [8] as an example of reference book on DG method). The idea is to solve problems defined in the weak form inside any mesh cell, without requiring the solution to be continuous at the mesh interfaces. At the interface, the fluxes are computed using standard Riemann solvers, as in Finite Volume – FV – formalism. Therefore, the FV flux computation enables the coupling of the weak problems in surrounding cells and the FV fluxes make information going across mesh interfaces. Several alternative high-order methods have been recently introduced. Following the staggered-grid multidomain spectral method [9] for structured grids, Liu, Vinokur and Wang [10,11] introduced the Spectral Difference – SD – method aiming at a simpler to implement and more efficient method than the current state of the art for the DG method. The approach was then extended to mixed elements [12]. The SD method takes benefit of the resolution of the strong differential form of the equations, as in FD, but does not assume that the solution is continuous on the whole mesh, as in FV. Another way to define a high-order polynomial reconstruction follows the definition of averaged quantities, as in FV. With the Spectral Volume – SV – approach [13–17], a polynomial reconstruction is defined inside any cell using the averaged quantities over sub-cells built by subdivision of the initial mesh elements. As before, several Riemann problems are solved on mesh boundaries since the solution polynomials are not required to be continuous at mesh interfaces. Finally, the Flux Reconstruction method introduced in 2007 by Huynh [18] solves the strong form of the equation. It can be seen as a collocated Spectral Difference scheme but the main difference occurs in the definition of the flux polynomial: now, a lifting operator [19–22] is introduced to increase the polynomial degree of the initial flux polynomial by one. This is mandatory to recover the required polynomial degree after the computation of the divergence (hyperbolic) term. The main advantage of FR method is its ability to recover SD, SV and DG approaches for the linear advection equation, depending on the lifting operator [23]. Compared to the standard schemes for structured grids, the major advantage of DG, SV, SD and FR methods lies in their natural ability to handle unstructured meshes, which is a prerequisite to treat complex geometries. Moreover, such schemes use a very compact stencil, defined locally inside any mesh cell. This is also an advantage in terms of high performance computing required by massively-parallel LES.

When dealing with aeroacoustics, the first question to answer concerns the spectral accuracy of the chosen scheme: the key point concerns the required number of grid points per wavelength. The spectral analysis [24] consists of dealing with the space derivative and in comparing the numerical spatial derivative with the theoretical derivative, after projection onto the Fourier basis. This analysis is performed on the linear advection equation in a periodic domain with a harmonic initial solution:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial t} + c\,\mathbb{D}(u) = 0 \\ u_0(x) = u(0,x) = \exp(jkx) \text{ with } j^2 = -1, \end{cases} \tag{1}$$

where the function $u(x,t)$ is the unknown, c the constant advective velocity, k the constant wavenumber and $\mathbb{D}$ represents the spatial derivative operator. As a consequence, Eq. (1) will play an important role and for sake of clarity, the notation for the unknowns will be kept unchanged along the whole document.

The formulation of the Fourier spectral analysis makes the analysis simple for standard schemes based on Finite Difference paradigm. This is due to the fact that the degrees of freedom are coupled by the numerics and not by the method itself. As introduced by Hu et al. in 1999 [25] for the DG method, the situation is more complex for spectral discontinuous methods. For DG method, the authors show that the degrees of freedom are coupled by the definition of the local polynomial – inside any mesh cell. Finally, the Fourier analysis can be performed as for FD approach but the final equation changes. Instead of one equation giving the complex-valued numerical wavenumber, one obtains, even for a scalar equation,