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Frame expansions with probabilistic erasures

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ABSTRACT

In modern communication network such as a public Internet, random losses of information can be recovered by frame expansions. In this paper, we think about the transport protocol that delivers a coefficient of frame expansions with some probability of failure via multiple intermediate nodes. By this protocol, we abstract the transmission as a Markov process. In this case, even if the probabilities of erasures are close to the 1/2, we show that the reconstruction error can be minimized as small as possible by using frames with high redundancy. Furthermore, we set up a probability model for constructing optimal Parseval frames that are robust to one probabilistic erasure. It is shown that probabilistic modeled Parseval frames are more effective than traditional Parseval frames on recovering the signal with probabilistic erasures.

1. Introduction

Frame, a redundant set of vectors in Hilbert space, was first introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series [1]. Compared to orthonormal bases, frames allow redundancy which is desirable in signal transmission for constructing signals when some coefficients are lost. Redundancy is one of the key features of frames that is important for applications. First, redundancy can provide us the flexibility for constructing frames that fit a particular problem in a manner not possible by a set of linearly independent vectors. The second advantage of redundancy is robustness. For example in the case that some of the coefficients of the encoded signal vectors have been intermittently faded out or lost (erased) in a transmission process, the redundancy of a frame used for coding can achieve resilience against losses (erasures) or noises. Hence, in contrast to orthonormal bases, frames can provide perfect reconstruction (or with much smaller reconstruction errors) of the signal from its erasure-corrupted data. Recently, frame theory and its applications have become an active area of research in both mathematics and engineering such as wireless communications [2], sampling theory [3], coding theory [4], filter bank [5] and image processing [6].

In a signal communication system, one can view this communication scheme with frame expansions as follows (see Fig. 1). The original signal is viewed as a vector $f \in \mathbb{R}^n$. This vector is represented by its $m \geq n$ expansion coefficients with respect to some given frames. These coefficients are sent through many multiple intermediate nodes. In this transmission process, erasures of expansion coefficients maybe arise from buffer overflows at the intermediate routers, or from global conditions in the network such as congestion, transmitted power. In this case, we may abstract the behavior of the network as delivering a coefficient with some probabilistic regularity of failure [7]. For instance, the probabilistic of bad channel failure is usually larger than the probability of good channel failure, or probability of erasure tends to increase with transmission time. In this case, the second author of this paper and other authors inset different weights for the expansion coefficients according to their degree of loss possibility. And they found optimal dual frames or optimal Parseval frames that will minimize the decoding errors when probabilistic erasures occur [8,9].

Compared to basis representation, a frame expansion can be a useful representation even when some frame coefficients are lost with some probabilistic regularity in transmission because proper subsets of frames are sometimes themselves frames. For example, it is not necessary to use all the coefficients to reconstruct the signal. A subset of the coefficients is sufficient to represent the signal as long as the corresponding frame elements still span the space. In this case, perfect reconstruction is possible, making the representation robust to probabilistic erasures during transmission. More importantly, authors of [9] showed that it is allowed to construct various frames in a flexible way for different application scenarios. For example, in a wireless relay network, every encoded data is delivered from one node to another. Due to the influence of external environments, the data is delivered with some probability of failure. When the loss probability of the data is obviously larger than that of the others, one can construct corresponding Parseval

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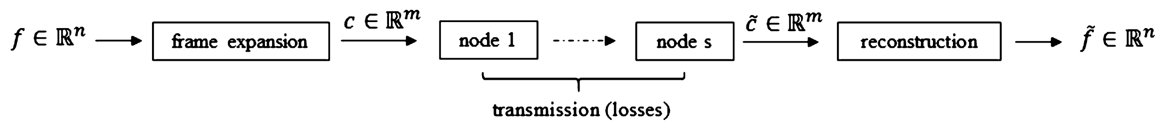

Fig. 1. The modern transmission model of a communication system using redundant signal expansion.

frames which are robust to probabilistic erasures. It is shown that this kind of Parseval frames perform better than traditional Parseval frames on recovering signals with erasures.

Based on above work, in this paper, we continue to study the work of minimizing the maximal error when the probabilistic erasures occur. We first analyze the transmission process and view it as a Markov process. Then we obtain the erasure probabilities of coefficients by a Markov transition matrix. We prove that there exists a frame which can recover the signal with a high degree of accuracy even if the erasure probabilities of coefficients are close to the 1/2. Then we define a new sequence of weights which is different from the one of [9]. We study the properties of weights which we have defined. It is shown that there exists a Parseval frame with the norm associated with weights, and we prove that this Parseval frame is optimal robust to probabilistic erasure. We remark that we neglect the quantization and other interference in this paper.

The paper is organized as follows. We recall some notations of frame theory and the process of frame expansion for coding in Section 2. In Section 3, we model the transmission of expansion coefficients as a Markov process. We prove that the error of the reconstructed signal can be minimized as small as possible by using a uniform tight frame, even if the each erasure probability of the coefficient is close to the 1/2. Section 4 is devoted to studying the weights which are defined by probabilities of erasures. It is shown that there exists a Parseval frame with norm associated with these weights. We call this Parseval frame a probabilistic modeled (PM) Parseval frame. We also prove that these Parseval frames are more effective than traditional Parseval frames on recovering the signal with probabilistic erasures. In Section 5, we present two numerical experiments to demonstrate our results. It is shown that the reconstruction errors become littler and littler as the redundancy of frames increase. And the second example shows that the newly proposed PM Parseval frames perform better than existing PM and traditional Parseval frames for recovering signals.

2. Preliminaries and frame expansion for coding

We denote by $\mathcal{H}$ a Hilbert space, by f^T the transposition of $f \in \mathcal{H}$ and by $\mathbb{E}(X)$ the expectation of a random variable X. $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. I is a countable index set. $|I|$ denotes the cardinality of a set I.

A sequence $\{f_i\}_{i\in I}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

The constants A and B are called *frame bounds*. If $A = B$ we call this frame an *A-tight frame* and if $A = 1$ it is called a *Parseval frame*. A *uniform frame* is a frame when all the elements in the frame sequence have the same norm. An *equiangular frame* is a frame if it satisfies the condition that $|\langle f_i, f_j\rangle|$ is a constant for all $i \neq j$.

An orthonormal basis is a Parseval frame, but not the converse. We recall an example of uniform equiangular Parseval frame which is not a basis: the Mercedes-Benz frame F in $\mathbb{R}^2$.

$$f_1 = \sqrt{\frac{2}{3}}[0,1]^T, \quad f_2 = \sqrt{\frac{2}{3}}\left[\frac{\sqrt{3}}{2}, -\frac{1}{2}\right]^T,$$

$$f_3 = \sqrt{\frac{2}{3}}\left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right]^T.$$

Let $F = \{f_i\}_{i\in I}$ be a frame for Hilbert space $\mathcal{H}$. The operator $\Theta_F : \mathcal{H} \to \ell^2(I)$ defined by

$$\Theta_F(f) = \sum_{i\in I} \langle f, f_i\rangle e_i$$

is called the *analysis (encoding) operator* of F, where $\{e_i\}_{i\in I}$ is the standard orthonormal basis for $\ell^2(I)$. A simple calculation shows that the adjoint operator Θ_F^* of Θ_F satisfies

$$\Theta_F^*\left(\sum_{i\in I} c_i e_i\right) = \sum_{i\in I} c_i f_i$$

and the operator Θ_F^* is called the *synthesis (decoding) operator* of F. Let $S_F = \Theta_F^*\Theta_F$, then we have

$$S_F(f) = \sum_{i\in I} \langle f, f_i\rangle f_i, \quad \forall f \in \mathcal{H}.$$

And S_F is called the *frame operator* which is a positive invertible bounded linear operator on $\mathcal{H}$. The following *reconstruction formula* holds:

$$f = \sum_{i\in I} \langle f, f_i\rangle S_F^{-1} f_i = \sum_{i\in I} \left\langle f, S_F^{-1} f_i\right\rangle f_i, \quad \forall f \in \mathcal{H}. \tag{1}$$

Then the family $\{S_F^{-1} f_i\}_{i\in I}$ is called the *canonical dual frame* of F.

If $F = \{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then the family $F^\circ = \{f_i^\circ\}_{i\in I}$, where $f_i^\circ = S_F^{-1/2} f_i$, $i \in I$, is a Parseval frame for $\mathcal{H}$.

The frame $\widetilde{F} = \{\widetilde{f}_i\}_{i\in I}$ is called an *alternate dual frame* of F if the following formula holds:

$$f = \sum_{i\in I} \langle f, f_i\rangle \widetilde{f}_i = \sum_{i\in I} \langle f, \widetilde{f}_i\rangle f_i, \quad \forall f \in \mathcal{H}. \tag{2}$$

For a more extensive introduction to frame theory, we refer the interested readers to the references [10–13].

For the purpose of coding, we only handle frames with finitely many vectors in finite dimension Hilbert space $\mathcal{H}$. Let $F = \{f_i\}_{i\in I}$ be a finite frame for an n-dimension Hilbert space $\mathcal{H}$ and $|I| = m$, then we necessarily have $m \ge n$. When $m = n$, F is a basis of $\mathcal{H}$. In order to ensure the redundancy for coding, we set $m > n$ throughout this paper. We also use the terminology (m, n)-frame to refer to a frame of m-elements for an n-dimensional Hilbert space $\mathcal{H}$.

In coding theory, a signal vector f is encoded as frame coefficients $\Theta_F(f) = \{\langle f, f_i\rangle\}_{i\in I}$. Then these coefficients are transmitted to a receiver for decoding to reconstruction the signal f. If the communication channel is perfect, so that no erasure occurs, then the signal can be perfectly reconstructed by using (1) (or (2)). But in a more realistic setting where the channel is not perfect, some coefficients may be erased during the transmission. In this case, recently many researchers have been working on different approaches to this problem.

Let $c_i = \langle f, f_i\rangle$ for all $i \in I$. Assume that $\{c_i\}_{i\in\Lambda^c}$ are the erased data in the transmission process, where $\Lambda \subset I$ and $\Lambda^c = I \setminus \Lambda$. One approach is to approximate f by (2), and the error operator is given by