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Adaptive mesh refinement method for optimal control using nonsmoothness detection and mesh size reduction

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Abstract

An adaptive mesh refinement method for solving optimal control problems is developed. The method employs orthogonal collocation at Legendre–Gauss–Radau points, and adjusts both the mesh size and the degree of the approximating polynomials in the refinement process. A previously derived convergence rate is used to guide the refinement process. The method brackets discontinuities and improves solution accuracy by checking for large increases in higher-order derivatives of the state. In regions between discontinuities, where the solution is smooth, the error in the approximation is reduced by increasing the degree of the approximating polynomial. On mesh intervals where the error tolerance has been met, mesh density may be reduced either by merging adjacent mesh intervals or lowering the degree of the approximating polynomial. Finally, the method is demonstrated on two examples from the open literature and its performance is compared against a previously developed adaptive method.

1. Introduction

Over the past two decades, direct collocation methods have become popular in the numerical solution of nonlinear optimal control problems. In a direct collocation method, the state and the control are discretized at a set of appropriately chosen points in the time interval of interest. The continuous-time optimal control problem is then transcribed to a finite-dimensional nonlinear

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programming problem (NLP) and the NLP is solved using well known software [1,2]. Originally, direct collocation methods were developed as *h* methods (for example, Euler or Runge–Kutta methods) where the time interval is divided into a mesh and the state is approximated using the same fixed-degree polynomial in each mesh interval. Convergence in an *h* method is then achieved by increasing the number and placement of the mesh points [3–5]. More recently, a great deal of research has been done in the class of direct *Gaussian quadrature orthogonal collocation* methods [6–23]. In a Gaussian quadrature collocation method, the state is typically approximated using a Lagrange polynomial where the support points of the Lagrange polynomial are chosen to be points associated with a Gaussian quadrature. Originally, Gaussian quadrature collocation methods were implemented as *p* methods using a single interval. Convergence of the *p* method was then achieved by increasing the degree of the polynomial approximation. For problems whose solutions are smooth and well-behaved, a Gaussian quadrature collocation method has a simple structure and converges at an exponential rate [24–26]. The most well developed Gaussian quadrature methods are those that employ either Legendre–Gauss (LG) points [10,15], Legendre–Gauss–Radau (LGR) points [16,17,19], or Legendre–Gauss–Lobatto (LGL) points [6].

Many mesh refinement methods employing *h* or *p* direct collocation methods have been developed previously. Reference [27] describes what is essentially a *p* method where a differentiation matrix is used to identify switches, kinks, corners, and other discontinuities in the solution. References [28,29] locally refine the grids by splitting selected intervals according to some splitting criterion. Reference [5] develops a fixed-order method that uses a density function to generate a sequence of non-decreasing size meshes on which to solve the optimal control problem. References [30,31] (and the references therein) describe a dual weighted residual (DWR) method for mesh refinement and goal-oriented model reduction. The DWR method uses estimates of a dual multiplier together with local estimates of the residuals to adaptively refine a mesh and control the error in problems governed by partial differential equations. Finally, in Ref. [3] an error estimate is developed by integrating the difference between an interpolation of the time derivative of the state and the right-hand side of the dynamics. The error estimate developed in Ref. [3] is predicated on the use of a fixed-order method (for example, trapezoid, Hermite–Simpson, Runge–Kutta) and computes a low-order approximation of the integral of the aforementioned difference. Different from all of this previous research where the order of the method is fixed and the mesh can only increase in size, in the method of this paper varies the degree of the polynomial approximation and the mesh size can be reduced.

While *h* methods have a long history and *p* methods have shown promise in certain types of problems, both the *h* and *p* approaches have limitations. Specifically, achieving a desired accuracy tolerance may require an extremely fine mesh (in the case of an *h* method) or may require the use of an unreasonably large degree polynomial approximation (in the case of a *p* method). In order to reduce significantly the size of the finite-dimensional approximation, and thus improve computational efficiency of solving the NLP, *hp* collocation methods have been developed. In an *hp* method, both the number of mesh intervals and the degree of the approximating polynomial within each mesh interval are allowed to vary. Originally, *hp* methods were developed as finite-element methods for solving partial differential [32–36]. In the past few years the problem of developing *hp* methods for solving optimal control problems has been of interest [20,21,23]. References [20,21] describe *hp* adaptive methods where the error estimate is based on the difference between an approximation of the time derivative of the state and the right-hand side of the dynamics midway between the collocation points. It is noted that the approach of Refs. [20,21] creates a great deal of noise in the error estimate, thereby making these