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On the quantification and efficient propagation of imprecise probabilities resulting from small datasets

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ABSTRACT

This paper addresses the problem of uncertainty quantification and propagation when data for characterizing probability distributions are scarce. We propose a methodology wherein the full uncertainty associated with probability model form and parameter estimation are retained and efficiently propagated. This is achieved by applying the information-theoretic multimodel inference method to identify plausible candidate probability densities and associated probabilities that each method is the best model in the Kullback-Leibler sense. The joint parameter densities for each plausible model are then estimated using Bayes' rule. We then propagate this full set of probability models by estimating an optimal importance sampling density that is representative of all plausible models, propagating this density, and reweighting the samples according to each of the candidate probability models. This is in contrast with conventional methods that try to identify a single probability model that encapsulates the full uncertainty caused by lack of data and consequently underestimate uncertainty. The result is a complete probabilistic description of both aleatory and epistemic uncertainty achieved with several orders of magnitude reduction in computational cost. It is shown how the model can be updated to adaptively accommodate added data and added candidate probability models. The method is applied for uncertainty analysis of plate buckling strength where it is demonstrated how dataset size affects the confidence (or lack thereof) we can place in statistical estimates of response when data are lacking.

1. Introduction

Uncertainty quantification (UQ) and propagation play a critical role in computationally evaluating the performance of engineering systems. Generally speaking, uncertainty can be categorized as either *aleatory*, resulting from inherent randomness, or *epistemic*, resulting from a lack of complete knowledge [1]. It has been argued that epistemic uncertainties require a different mathematical treatment than aleatory uncertainties that are naturally stochastic [2]. It remains an open, and sometimes contentious, debate as to what that mathematical treatment should be.

Arguments have been made for a variety of probabilistic and non-probabilistic treatments of epistemic uncertainties. Non-probabilistic uncertainty theories include Dempster-Schafer evidence theory [3,4], fuzzy sets [5], interval methods [6,7], convex models [8], possibility theory [9], and information theory [10] among others. Probabilistic approaches meanwhile include methods based on random sets [11], probability boxes (p-boxes) [12,13], Bayesian [14], and frequentist [15] theories. Moreover, several variations of and connections between both probabilistic and non-probabilistic theories have

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been made [16–18] in the interest of constructing a unified theory of imprecise probabilities [19,20]. In other words, to think of these theories as necessarily distinct and mutually exclusive is incorrect. Common formalisms can be applied across many of these approaches such that, for example, fuzzy sets can be interpreted as generalizations of both random sets and intervals and can also be interpreted to loosen the conditions on probability measures. For the interested reader, an extensive review of many of these approaches for engineering applications can be found in [21].

While the intricacies, interconnections, and nuances of these theories are critical to a comprehensive understanding of uncertainty, this work is motivated by epistemic uncertainty that specifically stems from having only small datasets available to characterize otherwise aleatory uncertainties. This is a problem that has been studied by several researchers (e.g. in the context of reliability analysis [22–24]), and one for which we propose a probabilistic approach that addresses its primary challenges: namely probability model selection, model parameter uncertainties, and uncertainty propagation. For clarity, herein the term *model* is used synonymously with a probability model, which more specifically assigns a probability measure to the probability space of the underlying aleatory uncertainty. Practically speaking, this is uncertainty in the form and parameter values of a probability distribution.

2. Uncertainty caused by lack of data

Statistical theories, both frequentist [15] and Bayesian [14], are well-suited for problems with large sample size. However, large datasets are rarely available for many engineering applications. Small sample size creates a specific form of epistemic uncertainty (sometimes referred to as second-order uncertainty [25]) that manifests in the inability to identify a unique model for the underlying probabilities. This section briefly discusses the challenges created by this type of uncertainty.

2.1. Model-form uncertainty

The desire to assign a specific model form (i.e. probability distribution type) is understandable from many perspectives. Certainly, it simplifies the uncertainty analysis. Going further, the inability to assign a model-form instills a sense of uneasiness in an analysts' mind because, among other reasons, tools for propagating multiple probability models are limited and computationally very expensive [25]. But one must always ask whether the assignment of a probability model from a small data set is justified. Undoubtedly, cases exist where assignment is justified such as when the data in the small set arise from, say, the sum of many other random variables such that the Central Limit Theorem applies. More commonly, distributions are assigned arbitrarily based on either analyst judgment or through some comparative down-selection or averaging process [26].

Through a comparative down-selection process, one begins with a number of candidate models and through statistical analysis selects the model that is "best" according to some criterion. The most commonly employed methods include statistical goodness-of-fit tests (e.g. [27]), Bayesian hypothesis testing and model selection [28,14], and information-theoretic model selection [29]. For small datasets, statistical goodness-of-fit tests are known to be inappropriate [30]. Bayesian and information-theoretic approaches, meanwhile, remain widely used when data is scarce (justifiably or not). The two methods differ in the nature of their selection criteria. Bayesian model selection methods use ratios of the integrated likelihood functions for different models to compute the Bayes factor. The information-theoretic process, on the other hand, bases the selection process on the Kullback-Leibler mean information loss. These methods are further elaborated in Section 3.

Another popular process is to average the various models that are being considered by assigning relative weights to them. This process, elaborated nicely for the lack of data uncertainties in [31], is perhaps a more comprehensive means of model selection in that it aims to include each of the possible candidate models through the averaging process.

Our primary disagreement with both the down-selection and averaging methods is not a methodological one, but a philosophical one. Given lack of data, we believe and attempt to show that the goal of identifying a single probability model capable of including both stochastic and small dataset uncertainty is an unrealistic one. Selecting a single model (averaged or not) disregards a portion of the uncertainty, and for this reason we prefer a multimodel inference approach, such as that presented by Burnham and Anderson [32] and outlined in Section 3.

2.2. Parameter uncertainty

For a given model, estimation of the model parameters introduces additional uncertainties. In the frequentist approach, the parameters are estimated as deterministic values, perhaps with confidence bounds established through methods such as bootstrapping [33]. Again, these methods can be problematic for small datasets. In the Bayesian approach, the parameters are treated as random variables with their joint distribution estimated through Bayes' rule as discussed in Section 3.1. For small datasets, these joint distributions can be heavily biased by the prior model and even the use of so-called noninformative priors can have important implications that will not be explored in more detail here.

In this work, we adopt the Bayesian approach with a noninformative prior (accepting that this perhaps has unintended consequences). The primary difference between the approach presented here and most Bayesian approaches is that, rather than estimating the joint parameter density and promptly discarding it by selecting a single set of maximum likelihood parameters for the model or integrating out its variability [34], we retain the full joint parameter densities for each model and propagate them through the function along with their parent models from the multimodel inference.