Mechanical Systems and Signal Processing 84 (2017) 34–53

Contents lists available at ScienceDirect

Mechanical Systems and Signal Processing

journal homepage: www.elsevier.com/locate/ymssp

A machine learning approach to nonlinear modal analysis

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ARTICLE INFO

Article history:
Received 8 August 2015
Received in revised form
13 April 2016
Accepted 24 April 2016
Available online 7 May 2016

Keywords:
Nonlinear modal analysis
Machine learning
Data-based analysis
Self-Adaptive Differential Evolution

ABSTRACT

Although linear modal analysis has proved itself to be the method of choice for the analysis of linear dynamic structures, its extension to nonlinear structures has proved to be a problem. A number of competing viewpoints on nonlinear modal analysis have emerged, each of which preserves a subset of the properties of the original linear theory. From the geometrical point of view, one can argue that the invariant manifold approach of Shaw and Pierre is the most natural generalisation. However, the Shaw–Pierre approach is rather demanding technically, depending as it does on the analytical construction of a mapping between spaces, which maps physical coordinates into invariant manifolds spanned by independent subsets of variables. The objective of the current paper is to demonstrate a data-based approach motivated by Shaw–Pierre method which exploits the idea of statistical independence to optimise a parametric form of the mapping. The approach can also be regarded as a generalisation of the Principal Orthogonal Decomposition (POD). A machine learning approach to inversion of the modal transformation is presented, based on the use of Gaussian processes, and this is equivalent to a nonlinear form of modal superposition. However, it is shown that issues can arise if the forward transformation is a polynomial and can thus have a multi-valued inverse. The overall approach is demonstrated using a number of case studies based on both simulated and experimental data.

1. Introduction

Modal Analysis is arguably *the* framework for structural dynamic testing of linear structures. While theoretical precursors existed before, the field really flourished in the 1970s with the advent of laboratory-based digital FFT analysers. The philosophy and main theoretical basis of the framework was encapsulated in the classic book [1] quite early and essentially stands to this day. The overall idea is to characterise structural dynamic systems in terms of a number of structural invariants: natural frequencies, dampings, mode shapes, and FRFs which can be computed from measured excitation and response data from the structure of interest.

While linear modal analysis has arguably reached its final form – although meaningful work remains to done in terms of parameter estimation etc. – no comprehensive nonlinear theory has yet emerged. Various forms of nonlinear modal analysis have been proposed over the years, but each can only preserve a subset of the desirable properties of the linear. Good

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http://dx.doi.org/10.1016/j.ymssp.2016.04.029

surveys of the state of the art at recent waypoints can be found in references [2,3].

Most of the theories of nonlinear modal analysis proposed so far depend on demanding algebra or detailed and intensive numerical computation based on equations of motion. The current paper proposes an approach based only on measured data from the system of interest and adopts a viewpoint, inspired by machine learning, of learning a transformation into 'normal modes' from the data.

The layout of the paper is as follows. Section 2 covers the main features of linear modal analysis, while Section 3 discusses how the approach breaks down for nonlinear systems. Section 4 gives a very condensed survey of some of the main approaches to nonlinear modal analysis taken in the past and takes some of the ideas presented as motivation for a new approach based on measured data. Examples of the new approach are presented in Section 5 via a number of case studies and the paper concludes with some overall discussion.

2. Linear systems

One begins here with the standard equation of motion for a Multi-Degree-of-Freedom (MDOF) linear system,

$$[m]\{\ddot{y}\} + [c]\{\dot{y}\} + [k]\{y\} = \{x(t)\} \tag{1}$$

where $\{x(t)\}$ is an excitation force, $\{y\}$ is the corresponding displacement response of the system and $[m]$, $[c]$ and $[k]$ are, respectively, the mass, damping and stiffness matrices of the system. (Throughout this paper, square brackets will denote matrices, curly brackets will denote vectors and overdots will indicated differentiation with respect to time.)

Linear modal analysis is predicated on the existence of a linear transformation of the responses,

$$[\Psi]\{u\} = \{y\} \tag{2}$$

such that, the equations of motion in the transformed coordinates,

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = [\Psi]^T\{x(t)\} = \{p\} \tag{3}$$

have diagonal mass, damping and stiffness matrices: $[M]$, $[C]$ and $[K]$. The matrix $[\Psi]$ is referred to as the *modal matrix*. The modal matrix encodes patterns of movement or coherent[1] motions of the structure. (It is well-known that matters are actually a little more complicated than this, e.g. complete diagonalisation of the matrices is only obtained if the damping matrix is proportional or of Rayleigh type, but the reader can consult basic texts for the details [1,4].) It immediately follows from Eq. (3) that all the degrees of freedom in the transformed coordinate system are uncoupled and each response $u_i(t)$ satisfies a Single-Degree-of-Freedom (SDOF) equation of motion,

$$m_i\ddot{u}_i + c_i\dot{u}_i + k_iu_i = p_i \tag{4}$$

where the m_i are referred to as *modal masses*. Corresponding to each of the new degrees of freedom are the standard natural frequencies and damping ratios,

$$\omega_{ni} = \sqrt{\frac{k_i}{m_i}}, \quad \zeta_i = \frac{c_i}{2\sqrt{m_ik_i}} \tag{5}$$

It is reasonable to refer to quantities like these as *modal invariants* because they are independent of the level of excitation. It is well-known that there is a dual frequency-domain representation of Eq. (4),

$$U_i(\omega) = G_i(\omega)P_i(\omega) \tag{6}$$

where $U_i(\omega)$ (resp. $P_i(\omega)$) is the Fourier transform of $u_i(t)$ (resp. $p_i(t)$) and $G_i(\omega)$ is a modal Frequency Response Function (FRF), defined by,

$$G_i(\omega) = \frac{1}{-m_i\omega^2 + ic_i\omega + k_i} \tag{7}$$

The overall structural FRF matrix $[H]$ defined by,

$$\{Y(\omega)\} = [H(\omega)]\{X(\omega)\} \tag{8}$$

can be recovered as a linear sum of the modal FRFs, i.e.

$$H_{ij}(\omega) = \sum_{k=1}^{N} \frac{\psi_{ik}\psi_{jk}}{-m_k\omega^2 + ic_k\omega + k_k} \tag{9}$$

[1] Modes are commonly referred to as periodic motions of the structure; however, this term seems not to capture the spatial synchronisation aspect of the mode shape. For this reason, the term *coherent motion* is preferred here.