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Support region estimation of the product polar companded quantizer for Gaussian source

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ABSTRACT

This paper describes two approaches to optimization of the key design parameters, the support region threshold and the number of magnitude representation levels, of product polar companded quantizer (PPCQ) for Gaussian source of unit variance. The first approach is based on the exact performance analysis of PPCQ and on distortion optimization with respect to the key design parameters. Due to the optimization problem complexity we encountered with the first approach, some suitable approximations are introduced with the second one. As a result, much simpler asymptotic closed-form formula for distortion of PPCQ is derived as a function of the support region threshold. Although with this approach the closed-form formula for the support region threshold cannot be derived, the results of this approach indicate the useful support region threshold form. By combining the results of both approaches we propose, the worthy closed-form formulas for the support region threshold and signal to quantization noise ratio of a nearly optimal PPCQ are provided. Moreover, from the results of both analyses the lower and upper bound expressions for the number of magnitude representation levels are provided. The analysis presented in the paper is useful for designing PPCQ and is of great theoretical and practical importance.

1. Introduction

Two-dimensional quantizer Q is every mapping from R^2 to the finite set of N points $\hat{Y} = \{\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \ldots, \hat{\mathbf{y}}_N\}$ from R^2. These points are known as the *representation levels*, while sets of all points $\mathbf{x} \in R^2$ such that $Q(\mathbf{x}) = \hat{\mathbf{y}}_i$ for a fixed i, are denoted by C_i and known as *quantization cells* [1]. We denote by R and Θ the magnitude and phase component of two-dimensional random variable $\mathbf{X} = (X_1, X_2)$. By radial symmetry, one can assume that the probability density function (PDF) $p_{X_1,X_2}(x_1, x_2)$ depends only on the signal magnitude $r = \|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$ and not on the phase component θ. Due to this symmetry, it is suitable to use the quantizer having its representation levels grouped on a certain number of circles. In other words, set of all representation levels, in polar coordinates, is given by $\hat{Y} = \{(\hat{r}_i, \hat{\theta}_{ij}),\ i = 1, 2, \ldots, L, j = 1, 2, \ldots, M_i\}$. Here, $\hat{r}_i$ are radii of circles and known as *radial representation levels*, $\hat{\theta}_{ij}$ are angular components of representation levels and known as *phase representation levels*, while L and M_i are total numbers of radial levels and phase levels corresponding to the ith radial level. In polar quantization, the total number of representation levels is $N = M_1 + M_2 + \cdots + M_L$. In the same time, quantization cells are given by [1] $C_{ij} = \{(r, \theta),\ r_{i-1} \le r < r_i, \theta_{ij-1} \le \theta < \theta_{ij}\}$ where r_i and θ_{ij} are *radial and phase region bounds*. Here $i = 0, 1, \ldots, L$, $j = 0, 1, 2, \ldots, M_i$, $r_0 = 0$, and $\theta_{i\,M_i} = \theta_{i\,0}$. Radial symmetry of the source further implies that the phase reconstruction levels $\hat{\theta}_{ij}$ and region bounds θ_{ij} are given uniformly, by [1–3] $\hat{\theta}_{ij} = 2\pi \cdot (2j - 1)/(2M_i)$, $j = 1, 2, \ldots, M_i$ and $\theta_{ij} = 2\pi \cdot j/M_i$, $j = 0, 1, 2, \ldots, M_i$.

The phase quantization can be dependent on the magnitude and then it is called *unrestricted polar quantization* [2,4–8]. Conversely, when the phase and magnitude are quantized independently, quantization is called *product polar quantization* or *restricted polar quantization* [9–14]. An unrestricted polar quantizer assigns a different number of phase levels at different magnitude levels. Unlike the unrestricted polar quantizer, a product polar quantizer has an equal number of phase reconstruction levels $M_i = M$ for every radial reconstruction level $\hat{r}_i$. The total number of representation levels for the product polar quantizer is equal to $N = LM$. Accordingly, as highlighted in [13] and [14], compared to unrestricted polar quantizers, product polar quantizers, we consider in this paper, are simpler from design and implementation point of view.

Companding technique is a commonly used way to design quantizers with large number of representation levels. This technique can be applied for scalar, vector and also, in the special case,

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for polar quantizers. In the case of polar companded quantization, representation levels and region bounds are given by [6–10]:

$$\hat{r}_i = g^{-1}((2i-1)/(2L)), \quad i = 1, 2, \ldots, L, \tag{1}$$

$$r_i = g^{-1}(i/L), \quad i = 0, 1, \ldots, L-1, \quad r_L = r_{max} = g^{-1}(1), \tag{2}$$

where g: $D \to [0, 1]$ is a certain function, known as *radial compressor function*. Additionally, g has to be continuous and growing function, having either $D = [0, +\infty)$ (infinite support for $r_{max} \to \infty$) or $D = [0, r_{max}]$ (finite support for a finite r_{max}) as its domain. For a finite support region threshold, r_{max}, the region $\{\mathbf{x} \in R^2 | \|\mathbf{x}\| \leq r_{max}\}$ is usually called a granular region, while its complement is named as an overload region [6,14].

Due to its significance, the problem of support region optimization has been extensively considered in the field of scalar quantization, for instance in [15–18]. However, possibly, due to the complexity reasons, the similar problem in the field of polar quantization has remained unexplored until recently. In particular, in [14] the optimization of the support region threshold and the number of magnitude levels of the simplest polar quantizer, i.e. product polar uniform quantizer, has been conducted and worthy closed-form formulas have been derived for the asymptotically optimal support region threshold of the magnitude quantizer and asymptotically optimal rate allocation between the magnitude and phase quantizers. Also, in the field of polar companded quantization, this problem has been recently analysed in [6,7]. Specifically, in [6] and [7], for the unrestricted polar companded quantizer designed asymptotically optimal for the Gaussian source, the finite support radial compressor function has been obtained by constraining the domain of the infinite support radial compressor function and then the optimization of the support region threshold has been performed so that the total asymptotic distortion per dimension is minimal. The optimization procedure performed in [6] and [7] have significantly contributed to the performances improvement of the unrestricted polar companded quantizer, especially noticeable for smaller bit rates. The results of [6,7] and [14], have motivated us to perform optimization of the product polar companded quantizer (PPCQ), which, to the best of the authors knowledge, has not been reported in the literature so far.

In the case of a finite support polar quantizer and radially symmetric source, quantizer performance, usually measured by *distortion*, is computed by summing the following expressions for the *granular distortion* and the *overload distortion* [1–6]:

$$D_g = \frac{1}{2} \sum_{i=1}^{L} \int_{r_{i-1}}^{r_i} \left[r^2 + \hat{r}_i^2 - 2r\,\hat{r}_i \; sinc(\pi/M)\right] p_R(r) dr, \tag{3}$$

$$D_{ol} = \frac{1}{2} \int_{r_{max}}^{+\infty} \left[r^2 + \hat{r}_L^2 - 2r\,\hat{r}_L \; sinc(\pi/M)\right] p_R(r) dr, \tag{4}$$

where $p_R(r)$ is the PDF of the magnitude component $R = \|\mathbf{X}\|$. Starting with Eqs. (3) and (4) and assuming finite support product polar companded quantization in this paper we derive the two formulas for distortion of the observed quantizer. The first one is exact formula and the other is an asymptotic formula for distortion represented as a function of the support region threshold r_{max} and the number of magnitude levels L. In order to minimize the distortion introduced by the model we observe, in this paper we perform optimization of both distortions with respect to r_{max} and L values. Specifically, we perform numerical optimization of the distortion determined by the exact formula, i.e. we use Simplex method and perform an exhaustive computer search for the optimal values of the key design parameters L and r_{max}. Since the optimization task is not a simple one, in this paper we also introduce some suitable approximations that provide the derivation of the asymptotic closed-form formulas for distortion of the observed quantizer facilitating the problem of optimization we have set. From the analytical expression for the support region threshold r_{max}, obtained as a result of asymptotic distortion optimization, the form of r_{max} dependence on the number of representation levels N is suggested. This dependence is particularly determined from the results of the previously mentioned exhaustive computer search. Specifically, we combine the results of two optimization approaches and we determine the closed-form formulas for r_{max} and SQNR as a function of the number of representation levels N by means of linear regression method. Moreover, based on the conducted analysis we provide the lower and upper bound formulas for the number of magnitude levels L.

The paper is organized as follows. Section 2 provides the exact analysis of the finite support product polar companded quantization, where fundamental expressions for granular D_g and overload distortion D_{ol} are derived. In Section 3 we perform an asymptotic analysis of the considered quantizer resulting in the derivation of the simple asymptotic expressions for granular and overload distortion as the function of quantizer design parameters L and r_{max}. We discuss the optimization procedure of design parameters, providing the expression for the asymptotically optimal value of L and asymptotic expression for r_{max}. Analysis is continued in Section 4 where the expressions are compared to the result of the exhaustive search for optimal parameters for $N = 32, 33, \ldots, 4096$. As a result, we propose the design procedure of nearly optimal PPCQ. The last section is devoted to the conclusions.

2. Exact analysis of finite support product polar companded quantizers

Assume that $\mathbf{X}$ is the two-dimensional Gaussian random variable with zero mean and unit variance. It is known that its magnitude and phase components (R and Θ) are distributed as follows [1]:

$$p_R(r) = re^{-r^2/2}, \quad r \geq 0, \qquad p_\Theta(\theta) = 1/(2\pi), \quad \theta \in [0, 2\pi]. \tag{5}$$

Expression for the granular distortion D_g of a finite support polar quantizer and radially symmetric source now became

$$D_g = \frac{1}{2} \sum_{i=1}^{L} \left[F_2(r_{i-1}, r_i) + \hat{r}_i^2 F_0(r_{i-1}, r_i) - 2\,\hat{r}_i \; sinc\left(\frac{\pi}{M}\right) F_1(r_{i-1}, r_i)\right], \tag{6}$$

where $F_i(r, r') = F_i(r') - F_i(r)$ and $F_i(r)$ $(i = 0, 1, 2)$ are incomplete moments of R given by

$$F_0(r) = \int_0^r p_R(\rho) d\rho = 1 - e^{-\frac{r^2}{2}}$$

$$F_1(r) = \int_0^r \rho\, p_R(\rho) d\rho = -re^{-\frac{r^2}{2}} + \sqrt{\frac{\pi}{2}} erf\left(\frac{r}{\sqrt{2}}\right). \tag{7}$$

$$F_2(r) = \int_0^r \rho^2 p_R(\rho) d\rho = 2 - e^{-\frac{r^2}{2}}(2 + r^2)$$

Note that $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the error function [19]. The overload distortion D_{ol} is computed in the similar manner

$$D_{ol} = \frac{1}{2} \sum_{i=1}^{L} \left[F_2(r_{max}, \infty) + \hat{r}_L^2 F_0(r_{max}, \infty) - 2\,\hat{r}_L \; sinc\left(\frac{\pi}{M}\right) F_1(r_{max}, \infty)\right], \tag{8}$$

where we end up with: