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Selection order framework algorithm for compressed sensing

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ABSTRACT

Greedy algorithms, which employ iterative greed strategy, are applied widely due to fast speed and simple structure. However, the reconstruction accuracy of greedy algorithm has a lot of room for improvement. To alleviate this drawback, an improved framework, called selection order framework, is proposed in this paper. This framework is very useful for greedy algorithms which use the correlation between columns of measurement matrix and the residue to select atoms per iteration. Moreover, to improve the recovery accuracy, the proposed framework only needs the selection order of atoms in estimated support set, which is available in original algorithm. The proposed framework also provides an adjustable parameter to control the tradeoff between the reconstruction accuracy and the run time. The efficiency of the proposed framework is demonstrated by simulations using sparse signals and a sparse image.

1. Introduction

Combining the signal sampling with compression for sparse signals, Compressed Sensing (CS) [4,10] theory can reduce the sampling rate and computational complexity of the measurement system. Given that the observed signal is sparse in a certain basis, CS can recover the original signal with a new sampling scheme that is much less than the conventional Shannon–Nyquist sampling rate. Based on this advantage, CS has many applications such as channel estimation [23], wireless sensor networks [16,26] and Magnetic Resonance Imaging [19,20,24,25].

However, reconstruction of the signal from the compressed measurements is NP-hard. Meanwhile, many different kinds of suboptimal solutions have been proposed to solve the problem in the literature. They can be broadly categorizes as: Greedy Pursuit, Convex Relaxation, and Bayesian framework. Among them, Greedy Pursuit algorithms have extensively application prospect by taking advantages of fast speed and simple structure. Representative algorithms are MP [17] and OMP [22], their improved algorithms, StOMP [11], SP [7], CoSaMP [18], SAMP [9], LAOMP [6] and FBP [15].

In order to improve the reconstruction accuracy of the greedy algorithm, many iterative [3] and fusion [1,2,8] methods are applied to the greedy algorithm. In [1], the author fuses two or more results to obtain a more accurate result. But this solution requires more than one algorithm to achieve this goal. In [3], the authors propose an iterative framework. The framework does not need other algorithms to improve the reconstruction accuracy. However, this framework requires the sparsity level as a priori information. Besides, the iteration termination condition determines that the frame is unable to adjust the tradeoff between the reconstruction accuracy and the run time. Moreover, the authors in [12–14] propose techniques of band exclusion (BE) and local optimization (LO) which can be embedded in the existing compressed sensing algorithms. They use those novel techniques to improve the algorithm performance.

The paper proposes the selection order framework (SOF). Just like the name of the framework, SOF exploits the selection order of chosen atoms to improve the performance of the parent algorithm. SOF does not need any other algorithm as assistant algorithm. Meanwhile, the sparsity level is also needless, if the parent algorithm does not need. Besides, SOF provides a parameter to control the maximum number of algorithm iterations, which can be used to adjust the compromise of the run time and reconstruction accuracy. Note that the exact recovery condition for all algorithms is selected as $||\mathbf{x}-\hat{\mathbf{x}}||_2 \leq 10^{-2}||\mathbf{x}||_2$ [15] in this paper.

2. Compressed sensing and reconstruction algorithm

2.1. Compressed sensing theory

Upon acquiring signals by the CS paradigm, we obtain the system of equations that are compactly represented by

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e}, \tag{1}$$

where $\mathbf{x}$ is a K-sparse signal of length N, K is the number of nonzero elements in $\mathbf{x}$, $\mathbf{b}$ is the observation vector of length M,

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and $\mathbf{A}$ is an $M \times N$ random matrix with $K < M < N$. $\mathbf{e}$ denotes the additive measurement noise of length M.

In [5], the authors defined the Kth restricted isometry constant δ_K of a matrix $\mathbf{A}$ as the smallest constant $\delta \geq 0$ such that

$$(1-\delta)||\mathbf{x}||_2^2 \leq ||\mathbf{Ax}|| \leq (1+\delta)||\mathbf{x}||_2^2 \quad \text{for all } K\text{-sparse}\mathbf{x} \in \mathbb{C}^{\mathbf{N}} \tag{2}$$

The correlation of any two columns $\boldsymbol{\varphi}_i$, $\boldsymbol{\varphi}_j$ of a matrix $\mathbf{A}$ can be expressed as:

$$\mu(i,j) := \mu(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) = \frac{|<\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j>_2|}{||\boldsymbol{\varphi}_i||_2||\boldsymbol{\varphi}_j||_2} \tag{3}$$

The mutual coherence [21] of $\mathbf{A}$ is the maximum pairwise coherence among all pairs of columns

$$\mu(\mathbf{A}) = \max_{i \neq j} \mu(i,j) \tag{4}$$

2.2. Reconstruction algorithm

In subsequent experiments, this paper would apply the proposed framework to these algorithms: OMP, SP and FBP. So we briefly introduce these three algorithms in this section.

According to the correlation value between the atoms and the residue vector of the last iteration, OMP selects one prominent atom per iteration. OMP can guarantee that an atom already selected will not be selected again by OMP in future iterations. In this work, the termination criterion is that the energy of the residue is less than the default threshold. Meanwhile, a parameter $K_{\max}$ is designed to prevent excessive iterations when OMP fails.

Different from OMP, SP selects the prominent K prominent atoms per iteration. This necessitates an a priori estimate of the sparsity level K. However, in most practical cases, K is unknown. The algorithm is iterated as long as the l_2-norm of the residue decreases.

FBP mainly contains two stages: forward step and backward step. By expanding and shrinking the estimated support set, FBP obtains a fixed number of atoms per iteration. The algorithm is iterated until the energy of the residue is less than the predefined threshold. Besides, FBP use the parameter $K_{\max}$ to avoid the algorithm running for too many iterations in case of a failure.

Algorithm 1 Modified OMP.

Input: $\mathbf{A}, \mathbf{b}, T^0, K_{\max}$;
Define: $\varepsilon, \mathbf{s} \in {}^{N\times 1}$;
Initialization:
$k = 0$;
$\mathbf{r}^o = \mathbf{b} - \mathbf{A}_{T^0}\mathbf{A}_{T^0}^{\dagger}\mathbf{b}$.
repeat:
1: $k = k + 1$;
2: $T_f = \arg\max_{J:|J|=1} ||\mathbf{A}_J^* \mathbf{r}^{k-1}||_1$;
3: $T^k = T^{k-1} \cup T_f$;
4: $\mathbf{s}_{T^k \backslash T^{k-1}}$ = the selection order of atom in $T^k \backslash T^{k-1}$
5: $\mathbf{r}^k = \mathbf{b} - \mathbf{A}_{T^k}\mathbf{A}_{T^k}^{\dagger}\mathbf{b}$
until $||\mathbf{r}^k||_2 \leq \varepsilon||\mathbf{b}||_2$ or $k \geq K_{\max}$
$\widetilde{\mathbf{x}} = 0, \widetilde{\mathbf{x}}_{T^k} = \mathbf{A}_{T^k}^{\dagger}\mathbf{b}$
Output: $\widetilde{\mathbf{x}}, T^k, \mathbf{s}$.

The pseudo codes of OMP, SP and FBP are respectively described in Algorithms 1–3. In order to adapt to the improved framework proposed in this paper, these algorithms have been slightly modified. The first change is adding an input parameter: the initial support set. The second change is adding an output arrays, which records the selection orders of atoms in the estimated support set. Step 4 in Algorithm 1, Step 6 in Algorithm 2 and Step 7 in Algorithm 3 are steps where output arrays record the selection orders of atoms. Once the new atom is determined, we would update the corresponding selection order in the array. We will introduce this initial support set and the selection orders of atoms in detail in Sections 3.2 and 3.3.

Algorithm 2 Modified SP.

Input: $\mathbf{A}, \mathbf{b}, T^0, K$;
Define: $\mathbf{s} \in {}^{N\times 1}, \mathbf{v} \in {}^{N\times 1}$;
Initialization:
$k = 0$;
$\mathbf{r}^o = \mathbf{b} - \mathbf{A}_{T^0}\mathbf{A}_{T^0}^{\dagger}\mathbf{b}$.
repeat:
1: $k = k + 1$;
2: $T_f = \arg\max_{J:|J|=1} ||\mathbf{A}_J^* \mathbf{r}^{k-1}||_1$;
3: $T^k = T^{k-1} \cup T_f$;
4: $\mathbf{v} = 0, \mathbf{v}_{\hat{T}^k} = \mathbf{A}_{\hat{T}^k}^{\dagger}\mathbf{b}$
5: $T^k = \arg\max_{J:|J|=K} ||\mathbf{v}_J||_1$
6: $\mathbf{s}_{T^k \backslash T^{k-1}}$ = the selection order of atom(s) in $T^k \backslash T^{k-1}$
7: $\mathbf{r}^k = \mathbf{b} - \mathbf{A}_{T^k}\mathbf{A}_{T^k}^{\dagger}\mathbf{b}$
until $||\mathbf{r}^k||_2 \geq ||\mathbf{r}^{k-1}||_2$
$\widetilde{\mathbf{x}} = 0, \widetilde{\mathbf{x}}_{T^{k-1}} = \mathbf{A}_{T^{k-1}}^{\dagger}\mathbf{b}$
Output: $\widetilde{\mathbf{x}}, T^{k-1}, \mathbf{s}$

Algorithm 3 Modified FBP.

Input: $\mathbf{A}, \mathbf{b}, T^0, K_{\max}$;
Define: $\alpha, \beta, \varepsilon, \mathbf{s} \in {}^{N\times 1}, \mathbf{v} \in {}^{N\times 1}$;
Initialization:
$k = 0$;
$\mathbf{r}^o = \mathbf{b} - \mathbf{A}_{T^0}\mathbf{A}_{T^0}^{\dagger}\mathbf{b}$.
repeat:
1: $k = k + 1$;
2: $T_f = \arg\max_{J:|J|=\alpha} ||\mathbf{A}_J^* \mathbf{r}^{k-1}||_1$
3: $T^k = T^{k-1} \cup T_f$;
4: $\mathbf{v} = \mathbf{0}, \mathbf{v}_{\hat{T}^k} = \mathbf{A}_{\hat{T}^k}^{\dagger}\mathbf{b}$
5: $T_b = \arg\min_{J:|J|=\beta} ||\mathbf{v}_J||_1$
6: $T^k = \hat{T}^k - T_b$
7: $\mathbf{s}_{T^k \backslash T^{k-1}}$ = the selection order of atom(s) in $T^k \backslash T^{k-1}$
8: $\mathbf{r}^k = \mathbf{b} - \mathbf{A}_{T^k}\mathbf{A}_{T^k}^{\dagger}\mathbf{b}$
until $||\mathbf{r}^k||_2 \leq \varepsilon||\mathbf{b}||_2$ or $|T^k| \geq K_{\max}$
$\widetilde{\mathbf{x}} = 0, \widetilde{\mathbf{x}}_{T^k} = \mathbf{A}_{T^k}^{\dagger}\mathbf{b}$
Output: $\widetilde{\mathbf{x}}, T^k, \mathbf{s}$

3. Selection order framework

3.1. The motivation of selection order framework

We will start with an experiment which shows the motivation of the proposed framework. Consider a system where the signal dimension N is 256, and the observation size M is 100. Meanwhile, we assume that the signal is a Gaussian sparse signal (see Section 4 for more details about the simulation setup). We consider the typical greedy algorithm OMP for reconstruction of the signal. In this example, we define a quality standard for the selec-