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Geometric multigrid technique for solving heat convection-diffusion and phase change problems

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ABSTRACT

Natural heat convection, mixed heat convection and heat transfer by conduction and convection with solidification of a ternary alloy are described by the finite volume method using a geometric multigrid approach. The objective of this paper is to analyze the effects of the multigrid technique on the accuracy and efficiency in describing convective heat transfer in closed and open cavities with and without liquid-solid phase changes of Newtonian and shear-thinning non-Newtonian fluids. It is found that the multigrid scheme reduces the computation time in natural convection in a square cavity from two times for Rayleigh number $Ra = 10^5$ up to seven times for $Ra = 10^3$, between 50% and 2.7 times for mixed convection with an inner solid when the Richardson number decreases from $Ri = 10$ to 0.1 and the Reynolds number $Re = 100$, and 20% for natural convection/heat conduction in solidification of a ternary aluminum alloy ($Ra = 10^4$) with a shear-thinning rheology and a power index equal to 0.5.

1. Introduction

Natural convective heat transfer is a relevant process present in many engineering applications such as air conditioning, temperature control of nuclear generators, melting and casting solidification, food freezing and thawing. Temperature gradients originate fluid motion activated by buoyancy forces driven by density changes. As a result, the coupling between fluid mechanics and natural convective heat transfer is stronger as the Grashof number increases. When the fluid is confined inside a cavity heat transfer is mainly by conduction at low Rayleigh numbers ($Ra < 10^3$), then by conduction and natural convection at intermediate values of Ra and finally by natural convection if $Ra > 10^5$. As the ratio between buoyancy and viscous forces increases, flow motion and heat transfer intensify and boundary layers of fluid mechanics and heat transfer become thinner close to the cavity walls. Therefore, accurate numerical solutions at high Rayleigh numbers must be found with a high number of nodes that increases the computational time.

The finite volume method is an efficient way to describe natural convection of incompressible flows [1–3]. The original approach to solve the discretized governing equations of continuity, linear momentum and energy introduced the SIMPLE prediction-correction segregated algorithm [4]. The iterative solution procedure was simplified by neglecting the velocity corrections of the neighboring nodes reducing the stability of the numerical solution and decreasing the convergence rate. These issues motivated the development of improved SIMPLE versions initiated by Patankar with SIMPLER [5], followed by SIMPLEC [6], SIMPLEX [7] and many others alternatives. A series of efficient pressure-correction algorithms: MSIMPLER [8], CLEAR [9], IDEAL [10] and CUT [11], have been developed by Tao and co-workers. Therefore, one attempt to reduce computing time with the finite volume method is to choose an accurate, robust, stable and fast predictor-corrector method.

An alternative approach to increase the convergence rate is to rely on geometrical [12] or algebraic multigrid methods [13]. Multigrid is a strategy that allows smoothing the residual of the discretized equations from a fine grid to the coarse one (restriction) followed by the interpolation of the update correction from the coarse grid to the fine one (prolongation) [14,15]. Multigrid schemes have been developed to increase the speed of convergence of compressible [16] and incompressible flows [17], including motion in heterogeneous porous media [18]. Multigrid techniques have been applied to describe laminar natural heat convection by the finite volume method with the SIMPLE algorithm [19]. The efficiency of the multigrid method in solving two dimensional steady forced convection of a laminar flow in a rectangular cavity with Reynolds numbers between 70 and 300 by the finite volume method with the SIMPLE algorithm has been reported by Mesquita and Lemos [20]. Eight levels of grids in a W multigrid cycle and a grid with

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512×512 nodes were used to increase the speed of convergence in the solution of steady two-dimensional natural heat convection in laminar flow in a square cavity for Rayleigh numbers Ra in the range between 10^2 and 10^7 [21].

The aim of this paper is to investigate the reduction in computation time when instead of the use of a single grid a three levels geometric multigrid technique in the finite volume method with the SIMPLE algorithms is implemented in the solution of three two-dimensional heat transfer problems. First, steady state natural convection in a square cavity is solved for Rayleigh numbers Ra = 10^3, 10^4, 10^5 and 10^6 and the effect of the grid size is investigated for two alternative restriction-prolongations cycles (V and W) of the multigrid method. The second case describes steady fluid mechanics and heat transfer of mixed heat convection in a lid-driven cavity with an inner hot solid for three Richardson numbers Ri = 0.1, 1 and 10 and two sizes of the internal solid block located at two different positions inside the cavity. Finally, the complex problem of transient natural convection and heat conduction with liquid-solid phase change of a ternary aluminum alloy with properties varying with temperature inside a square cavity is investigated for the cases of a Newtonian and a shear-thinning non-Newtonian rheology (power index n = 0.5) for both the molten alloy and the mushy zone. Accuracy of the numerical solutions for velocity, pressure and temperature obtained with multigrid is verified with results obtained with a single grid and with reliable numerical results reported for each problem.

2. Physical and mathematical models

Three natural heat convective problems in laminar flow of increasing complexity are investigated by numerical simulations with the finite volume method:

(1) Problem 1: steady natural convection in a square cavity;
(2) Problem 2: steady mixed convection in a lid driven cold cavity with an inner hot solid body;
(3) Problem 3: unsteady natural convection and conduction with solidification of a ternary alloy.

In problem 1 fluid and heat flows in a differentially heated cavity with a temperature difference applied to the vertical walls and horizontal adiabatic walls are investigated for Rayleigh numbers Ra = 10^3, 10^4, 10^5 and 10^6.

The presence of an inner solid at a high uniform temperature located inside a driven lid cavity with cold walls, in which the size and location of the solid block are changed, originates the mixed heat convection being analyzed in problem 2 for a Reynolds number Re = 100 and Richardson numbers Ri = 0.1, 1 and 10. Fig. 1 shows the physical models with the dimensionless boundary conditions imposed in problems 1 and 2.

The third problem consider unsteady heat natural convection and conduction with solidification of Al-7 wt%SiF.3 wt%Mn alloy inside a square cavity with sides equal to 0.024 m being cooled by heat convection on the left vertical wall (Biot number $Bi = 3.3$; $T_f = 25.5$ °C) while the three remaining walls are adiabatic (Fig. 2). Initially the molten alloy is at rest with a uniform temperature $T_0 = 860$ °C with the liquid-solid phase change starting with a *liquidus* temperature $T_L = 615$ °C and ending when a *solidus* temperature $T_S = 545$ °C is reached.

Problems 1 and 2 are based on natural convective heat transfer with constant properties but density and Newtonian fluids with unsteady mathematical models that can be built as simplified cases of the third one. Problem 3 includes the change of the physical properties with temperature, solidification with a liquid phase change fraction varying non-linearly with temperature and the cases of Newtonian and shear-thinning non-Newtonian rheology for the molten alloy and in the mushy zone. In the three cases density is assumed to change linearly with temperature and the Boussinesq approximation is imposed. The two-dimensional mathematical model of the third problem, considering a shear-thinning non-Newtonian rheology for the molten alloy and for the liquid-solid mixture in the mushy zone, including the variation of specific heat and thermal conductivity with temperature, is stated in terms of continuity, linear momentum and energy equations as follows [22]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\eta\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\,\frac{\partial u}{\partial y}\right) \quad (2)$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\eta\,\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\,\frac{\partial v}{\partial y}\right) + g\beta(T - T_m) \quad (3)$$

$$\rho\left[\frac{\partial(\overline{C_p}T)}{\partial t} + u\frac{\partial(\overline{C_p}T)}{\partial x} + v\frac{\partial(\overline{C_p}T)}{\partial y}\right] = \frac{\partial}{\partial x}\left(k\,\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial T}{\partial y}\right) \quad ; \quad \overline{C_p} = C_p + f_{pc}L \quad (4)$$

$$f_{pc} = \begin{cases} 0 \rightarrow T \le T_S \\ \frac{T - T_j}{T_{j+1} - T_j} \rightarrow T_j \le T \le T_{j+1} \quad ; \quad T_S < T < T_L \\ 1 \rightarrow T > T_L \end{cases} \quad (5)$$

$$\eta = \frac{\eta_l}{f_{pc}} \quad ; \quad \eta_l = \mu^* \dot{\gamma}^{n-1} \quad ; \quad \dot{\gamma} = \left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]^2\right\}^{1/2} \quad (6)$$

where the phase change liquid fraction (f_{pc}) used to calculate the effective viscosity in the mushy two phase zone is a non-linear function of temperature that increases the coupling between fluid mechanics and convective heat transfer.

The initial condition considers the molten alloy at rest and at uniform temperature: $u = v = 0$ and $T = T_0$ at $t = 0$, while the boundary conditions include no-slip conditions at the four walls and three adiabatic surfaces with a convective cooling imposed on the left vertical surface,

$$u = v = 0, \quad -k\frac{\partial T}{\partial x} = h(T - T_\infty), \quad \boldsymbol{at}\ \boldsymbol{x} = 0, \quad 0 \le \boldsymbol{y} \le \boldsymbol{H} \quad (7)$$

$$u = v = 0, \quad \frac{\partial T}{\partial x} = 0, \quad \boldsymbol{at}\ \boldsymbol{x} = \boldsymbol{L}, \quad 0 \le \boldsymbol{y} \le \boldsymbol{H} \quad ;$$
$$u = v = 0, \quad \frac{\partial T}{\partial y} = 0, \quad \boldsymbol{at}\ \boldsymbol{y} = 0, \boldsymbol{H}, \quad 0 \le \boldsymbol{x} \le \boldsymbol{L} \quad (8)$$

Table 1 presents the physical properties of the ternary alloy while the variations with temperature of the constant pressure specific heat (C_p) and thermal conductivity (k) are given in Table 2.

The phase change liquid fraction of Al-7%Si-0.3%Mn varies non-linearly with temperature as it was found from physical experiments [23]. Present numerical simulations were accomplished by using nine piece-wise linear interpolation functions of f_{pc} in terms of the temperature given by,

$$f_{pc} = a + bT;\ -102.359 < a < -4.364;\ 0.00625 < b < 0.18163 \quad (9)$$

where temperature intervals are those shown with dots in Fig. 3, and temperature is in °C.

3. Numerical procedure

In all the problems considered, the physical-mathematical model was solved by the finite volume method and the SIMPLE algorithm.