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# Two-sided thermal resistance estimates for heat transfer through an anisotropic solid of complex shape

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ABSTRACT

A mathematical heat transfer model in variational form is constructed for steady-state heat transfer through an inhomogeneous anisotropic solid of arbitrary configuration. This model is used to obtain two-sided thermal heat transfer resistance estimates for such a solid. These estimates permit calculating the maximum possible error when approximating the thermal resistance by half the sum of these estimates.



## 1. Introduction

The thermal heat transfer resistance between heating media separated by device elements is a quantity characteristic that is important when designing various heat exchange devices and analyzing their efficiency. Note that although most existing methods for heat transfer analysis only apply to isotropic materials, the structural elements of modern power plants and thermal protection systems also widely use anisotropic materials, including composites of various structure [1–5].

The thermal resistance of heat transfer through an inhomogeneous anisotropic solid between surface sections experiencing convective heat exchange with the ambient medium can be estimated with the use of a mathematical model describing the steady-state temperature distribution in the solid. A transformation of this mathematical model permits constructing a dual variational statement of the steady-state thermal conduction problem, which contains two functionals dual to each other (one to be minimized and the other to be maximized) [6–9] attaining the same extremal value on the actual solution of the problem. The desired thermal heat transfer resistance can be expressed via this value. However, finding the actual temperature distribution in an anisotropic solid of complex shape with an (in general) inhomogeneous surface distribution of the convective heat transfer coefficient is rather difficult even with the use of numerical methods and modern hardware [10,11].

The dual variational statement of the steady-state thermal conduction problem permits obtaining guaranteed successively tightening two-sided estimates of the desired heat transfer coefficient. These estimates may be sufficient for selecting the design and operational parameters of a heat exchange device at the first design stage. Since the values of the functional to be minimized on any admissible temperature distributions are not less than its value on the actual distribution, it follows that the former values provide a guaranteed upper bound for the actual value of the heat transfer coefficient. Likewise, the values of the functional to be maximized on any admissible distributions are not greater than its value on the actual distributions, which provides a lower bound for the actual value of the heat transfer coefficient. These two-sided estimates permit finding the maximum possible relative error in the approximate value of the heat transfer coefficient chosen when designing the heat exchange device.

## 2. Constitutive relations

Consider an inhomogeneous anisotropic solid occupying a volume $V$ bounded by a surface $S$ (Fig. 1). Convective heat exchange with the ambient medium at constant temperatures $T_1$ and $T_2$ occurs on parts $S_1 \subset S$ and $S_2 \subset S$, respectively, of the surface; these parts are assumed not to be in contact with each other. The convective heat exchange intensity is determined by the heat transfer coefficients $\alpha_1(P_1)$ and $\alpha_2(P_2)$, which depend on the position of

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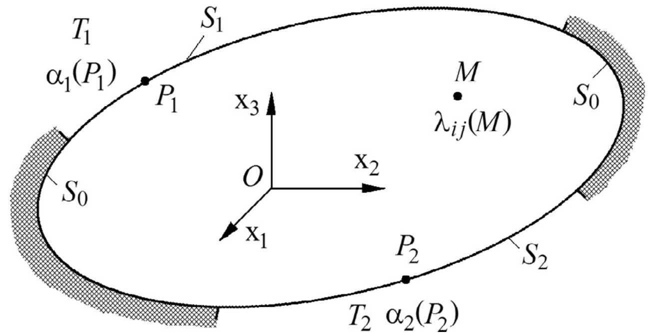


**Fig. 1.** Calculation scheme.

the points $P_1 \in S_1$ and $P_2 \in S_2$ of these parts. Heat exchange by radiation is not taken into account. The remaining part $S_0$ of the surface $S$ is perfectly thermally insulated. The thermal conductivity of the solid is characterized by a symmetric rank 2 tensor with temperature-independent components $\lambda_{ij}(M)$, $i,j = 1,2,3$, defined in the selected rectangular Cartesian coordinate system $Ox_1x_2x_3$ and depending on the coordinates of the point $M \in V$ in the volume $V$. This tensor can be reduced to principal axes, in which it has diagonal form with positive eigenvalues for real materials; i.e., the $3 \times 3$ symmetric matrix corresponding to this tensor is positive definite [10].

The steady-state temperature distribution $T(M)$ $(M \in \overline{V})$ in the closed volume $\overline{V} = V \cup S$ satisfies the differential equation

$$\frac{\partial}{\partial x_i}\left(\lambda_{ij}(M)\frac{\partial T(M)}{\partial x_j}\right) = 0, \quad M \in V, \tag{1}$$

and the boundary conditions

$$\lambda_{ij}(P_1)\frac{\partial T(P_1)}{\partial x_j}n_i(P_1) + \alpha_1(P_1)(T(P_1) - T_1) = 0, \quad P_1 \in S_1, \tag{2}$$

$$\lambda_{ij}(P_2)\frac{\partial T(P_2)}{\partial x_j}n_i(P_2) + \alpha_2(P_2)(T(P_2) - T_2) = 0, \quad P_2 \in S_2, \tag{3}$$

$$\lambda_{ij}(P_0)\frac{\partial T(P_0)}{\partial x_j}n_i(P_0) = 0, \quad P_0 \in S_0 = S \setminus (S_1 \cup S_2), \tag{4}$$

where the $n_i$ are the direction cosines of the outward normal to $S$ at the corresponding points of the surface.

Eq. (1) and the boundary conditions (2)–(4) compose a differential form of the mathematical model of a steady-state thermal conduction process in the solid in question.

## 3. Variational form of the mathematical model

Associated with the differential form (1)–(4) of the mathematical model of the steady-state thermal conduction process is the variational form, which contains a functional to be minimized. To construct a functional whose stationary point satisfies (1)–(4), multiply Eq. (1) by a variation $-\delta T(M)$ $(M \in V)$ and integrate over the volume $V$; multiply Eq. (2) by $\delta T(P_1)$ $(P_1 \in S_1)$ and integrate over the part $S_1$ of the surface $S$; multiply Eq. (3) by $\delta T(P_2)$ $(P_2 \in S_2)$ and integrate over $S_2$; multiply Eq. (4) by $\delta T(P_0)$ $(P_0 \in S_0)$ and integrate over $S_0$. By summing these integrals, we obtain (the arguments of the functions are omitted)

$$-\int_V \tfrac{\partial}{\partial x_i}\left(\lambda_{ij}\tfrac{\partial T}{\partial x_j}\right)\delta T dV + \int_{S_1}\left(\lambda_{ij}\tfrac{\partial T}{\partial x_j}n_i + \alpha_1(T - T_1)\right)\delta T dS$$
$$+\int_{S_2}\left(\lambda_{ij}\tfrac{\partial T}{\partial x_j}n_i + \alpha_2(T - T_2)\right)\delta T dS + \int_{S_0}\lambda_{ij}\tfrac{\partial T}{\partial x_j}n_i\delta T dS = 0.$$

Here we apply the Gauss divergence theorem to the first integral and represent the left-hand side as the variation $\delta J[T, \delta T]$ of the functional

$$J[T] = \int_V \frac{\partial T}{\partial x_i}\frac{\lambda_{ij}}{2}\frac{\partial T}{\partial x_j}dV + \int_{S_1}\alpha_1 T\frac{T - 2T_1}{2}dS + \int_{S_2}\alpha_2 T\frac{T - 2T_2}{2}dS. \tag{5}$$

The functional (5) can be considered on the set of temperature distributions $T(M)$, $M \in V$, that are continuous in $V$ and differentiable with respect to the spatial coordinates everywhere in $V$ possibly except on some set consisting of lines or surfaces. Admissible temperature distributions do not necessarily satisfy the boundary conditions (2)–(4), which are natural for this functional [12].

An analysis of the extremal properties of the functional (5) shows [13] that it is strictly convex on the entire set of admissible temperature distributions and attains its unique minimum at its stationary point on the actual temperature distributions $T^*(M)$ $(M \in \bar{V})$. Indeed, the integrands in the integrals over $S_1$ and $S_2$ are strictly convex. The integrand in the first integral on the right-hand side in (5) is strictly convex as well owing to the positive definiteness of the matrix corresponding to the thermal conductivity tensor if the temperature gradient is a nonzero vector. Otherwise, the zero value of this function does not affect the overall strict convexity of the functional.

The strict convexity of the functional (5), which is usually called the primal functional, implies the uniqueness of its stationary value, which is in turn equivalent to the uniqueness of the solution of the steady-state thermal conduction problem described by the mathematical model (1)–(4). By transforming the first integral in (5) by Green's first formula, which in our case has the form

$$\int_V \frac{\partial T}{\partial x_i}\lambda_{ij}\frac{\partial T}{\partial x_j}dV = \int_S T\lambda_{ij}\frac{\partial T}{\partial x_j}n_i dS - \int_V T\frac{\partial}{\partial x_i}\left(\lambda_{ij}\frac{\partial T}{\partial x_j}\right)dV,$$

one can show that the following value of the functional (5) corresponds to the actual temperature distributions satisfying the differential form (1)–(4) of the mathematical model:

$$J[T^*] = -\frac{T_1}{2}\int_{S_1}\alpha T^* dS - \frac{T_2}{2}\int_{S_2}\alpha T^* dS. \tag{6}$$

## 4. Dual variational form of the mathematical model

To construct a dual functional for (5), i.e., a functional whose maximum value coincides with $J[T^*]$ and is attained on the actual solution of the problem, let us extend the domain of the functional (5) by introducing a vector function $q(M)$, $M \in \overline{V}$, satisfying the condition

$$q_i(M) + \lambda_{ij}(M)\frac{\partial T(M)}{\partial x_j} = 0, \quad M \in \overline{V}, \tag{7}$$

where the $q_i$ are the projections of the heat flux density vector $\mathbf{q}$ onto the axes $Ox_i$. Let us eliminate the projections of the temperature gradient from the functional (5) by using condition (7) and simultaneously, with the help of the vector Lagrange multiplier $\mathbf{m}(M)$ defined at the points $M \in \overline{V}$ and having projections $m_i$ onto the axes $Ox_i$, introduce this condition in the functional (5). Thus, omitting the arguments of functions, we write

$$I_1[T, q_i, m_i] = \int_V q_i\frac{r_{ij}}{2}q_j dV + \int_{S_1}\alpha_1 T\frac{T - 2T_1}{2}dS$$
$$+\int_{S_2}\alpha_2 T\frac{T - 2T_2}{2}dS - \int_V m_i\left(q_i + \lambda_{ij}\frac{\partial T}{\partial x_j}\right)dV, \tag{8}$$