International Journal of Thermal Sciences 117 (2017) 68–76

Contents lists available at ScienceDirect

International Journal of Thermal Sciences

journal homepage: www.elsevier.com/locate/ijts

Estimation of linearly temperature-dependent thermal conductivity using an inverse analysis

Farzad Mohebbi [a,*], Mathieu Sellier [b], Timon Rabczuk [a]

[a] Institute of Structural Mechanics, Bauhaus-University Weimar, Marienstraße 15, 99423 Weimar, Germany
[b] Department of Mechanical Engineering, University of Canterbury, Private Bag 4800, Christchurch 8140, New Zealand

ARTICLE INFO

Article history:
Received 17 October 2016
Received in revised form
13 March 2017
Accepted 13 March 2017

Keywords:
Inverse heat transfer
Elliptic grid generation
Sensitivity analysis
Finite difference method
Conjugate gradient method
Variable thermal conductivity

ABSTRACT

A linearly temperature-dependent thermal conductivity is estimated in steady state heat conduction problems using an inverse analysis. A body fitted grid generation technique is employed to mesh the two-dimensional body and solve the direct heat conduction problem. An efficient, accurate, and easy to implement method is presented to compute the sensitivity coefficients through derived expressions. The main feature of the sensitivity analysis is that all sensitivities can be obtained in one solve, irrespective of the number of unknown parameters. The conjugate gradient method along with the discrepancy principle is used in the inverse analysis to minimize the objective function and achieve the desired solution. The ability to efficiently and accurately recover the non-constant thermal conductivity is demonstrated through a number of benchmark problems.

1. Introduction

Due to the increasing development of powerful computers in the past decades, the numerical treatment of inverse heat transfer problems (IHTP) has gained much attention. However, difficulties occur in the solution of such problems due to their ill-posed nature. Ill-posed problems are inherently unstable and very sensitive to error in the measurements used in the analysis. Among the methods to overcome the instabilities in inverse heat transfer problems are the iterative regularization methods [1]. There is no need to modify the original objective function in iterative regularization methods. In these gradient based methods, the discrepancy principle may be used as a criterion to terminate the iteration and obtain a reasonably stable solution. IHTP deals with the determination of the boundary conditions, the thermo-physical properties, the geometrical configuration of the heated body, and the heat flux by knowing the temperature distribution on some part of the heat conducting body boundary. Contrary to IHTP, the well-posed direct heat transfer problems are concerned with the determination of the temperature distribution in the body by having the boundary conditions, the thermo-physical properties, the body geometrical configuration, and the heat flux applied at some part of the body boundary [1,2]. The inverse analysis has been extensively employed in the estimation of the thermal conductivity and the heat transfer coefficient [3–32], the heat flux [16,33–39], and the determination of the boundary shape of bodies [40–46], to name a few.

There exist materials in which the thermal conductivity varies with the temperature. The heat conduction equation with variable thermal conductivity is a nonlinear equation and the numerical solution of this equation and the associated inverse analysis needs special consideration. In this study, a linearly temperature-dependent thermal conductivity is used in problems governed by the steady state heat conduction equation (with no heat generation). Although the linear form for this dependency is the simplest one, we can extend the method to other forms of dependency. The proposed procedure takes advantage of the two dimensional elliptic grid generation technique to mesh an irregular 2-D body, a nonlinear least square formulation to define the objective function, an efficient and accurate sensitivity analysis scheme to compute the sensitivity coefficients, and a gradient based optimization method. The conjugate gradient optimization method is used as a tool to reduce the mismatch between the estimated temperatures (obtained from the solution of the direct heat transfer problem) and the measured temperatures and the discrepancy principle is used to terminate the iterative procedure. The main feature of the

* Corresponding author.
E-mail address: farzadmohebbi@yahoo.com (F. Mohebbi).

http://dx.doi.org/10.1016/j.ijthermalsci.2017.03.016
1290-0729/

proposed method is that the sensitivity problem can be solved without requiring solving the adjoint problem. An explicit expression for the sensitivity coefficients is derived which allows us to compute the sensitivity coefficients in one single solution of the direct heat transfer problem (at each iteration), regardless of the number of unknown variables appearing in the inverse analysis. For many materials, the thermal conductivity can be approximated as a linear function of temperature over limited temperature ranges and expressed as $k(T) = k_0(1 + \beta T)$ where β is called the temperature coefficient of thermal conductivity [47]. In this study, therefore, the thermal conductivity is regarded as $k_T = a + bT$ where T is the temperature and a and b are constant ($b \neq 0$). As will be explained later, however, other forms of dependency of the thermal conductivity on the temperature, such as $k_T = a + bT + cT^2$ (quadratic) and $k_T = a + bT + cT^2 + dT^3$ (cubic), may also be used.

The numerical algorithm presented in this study is sufficiently general by which we can determine the variable thermal conductivity of a *general* two-dimensional region (heat conducting body) with *Neumann* and *Robin* conditions at the boundaries as long as the general two dimensional region can be mapped onto a regular computational domain.

2. Governing equation

The mathematical formulation for the steady state heat conduction problem with linearly temperature - dependent thermal conductivity is given by (see Fig. 1a)

$$\frac{\partial}{\partial x}\left(k_T\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_T\frac{\partial T}{\partial y}\right) = 0 \text{ in physical domain } \Omega \tag{1}$$

subject to the boundary conditions

$$\frac{\partial T}{\partial n} = \frac{\dot{q}}{k_T} \text{ on boundary surface } \Gamma_1 \tag{2}$$

$$\frac{\partial T}{\partial n} = -\frac{h_i}{k_T}\left(T_{\Gamma_i} - T_{\infty_i}\right) \text{ on boundary surface } \Gamma_i, i = 2,3,4 \tag{3}$$

where $k_T = a + bT$; a and b are constant and $b \neq 0$.

The elliptic grid generation method is employed here to discretize the physical domain and approximate the derivatives of the field variable (temperature) by algebraic ones. In this method, the irregular physical domain is mapped from the x and y physical plane onto the ξ and η computational plane (Fig. 1). Then the heat

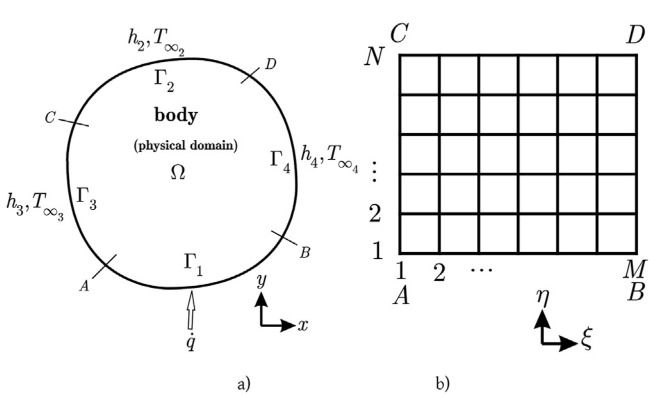

Fig. 1. Arbitrarily shaped two dimensional heat-conducting body (physical domain) subjected to convective heat transfer on surfaces $\Gamma_i, i = 2,3,4$ and heat flux $\dot{q}$ on surface Γ_1 (a) and the corresponding computational domain (b). The thermal conductivity of the body is k_T.

conduction equation and the boundary conditions (Eqs. (1) to (3)) should be transformed from the x and y physical plane to the ξ and η computational plane. More details on the implementation of the elliptic grid generation technique and solution procedure for the steady state heat conduction equation can be found in Ref. [48]. Here because the thermal conductivity is not constant and is linearly temperature dependent, we can expand Eq. (1) as follows

$$\frac{\partial}{\partial x}\left((a+bT)\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left((a+bT)\frac{\partial T}{\partial y}\right) = 0$$

$$(a+bT)\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + b\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right] = 0 \tag{4}$$

we can substitute for T_x, T_y, T_{xx}, and T_{yy}, using the transformation relationships and finite difference expressions [48]:

$$(a+bT)\left[\frac{1}{J^2}\left(\alpha T_{\xi\xi} - 2\beta T_{\xi\eta} + \gamma T_{\eta\eta}\right)\right] + b\left[\frac{1}{J^2}\left(y_\eta T_\xi - y_\xi T_\eta\right)^2 + \frac{1}{J^2}\left(-x_\eta T_\xi + x_\xi T_\eta\right)^2\right] = 0$$

$$(a+bT)\left[\left(\alpha T_{\xi\xi} - 2\beta T_{\xi\eta} + \gamma T_{\eta\eta}\right)\right] + b\left[\alpha T_\xi^2 - 2\beta T_\xi T_\eta + \gamma T_\eta^2\right] = 0$$

$$\begin{aligned}&(a+bT_{i,j})\Big(\alpha(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}) - 2\beta\Big(\frac{1}{4}T_{i+1,j+1} - \frac{1}{4}T_{i-1,j+1} \\ &- \frac{1}{4}T_{i+1,j-1} + \frac{1}{4}T_{i-1,j-1}\Big) + \gamma(T_{i,j+1} - 2T_{i,j} + T_{i,j-1})\Big) \\ &+ b\Big(\alpha\Big(\frac{1}{2}T_{i+1,j} - \frac{1}{2}T_{i-1,j}\Big)^2 - 2\beta\Big(\frac{1}{2}T_{i+1,j} - \frac{1}{2}T_{i-1,j}\Big)\Big(\frac{1}{2}T_{i,j+1} \\ &- \frac{1}{2}T_{i,j-1}\Big) + \gamma\Big(\frac{1}{2}T_{i,j+1} - \frac{1}{2}T_{i,j-1}\Big)^2\Big) \\ &= 0\end{aligned} \tag{5}$$

where

$$\alpha = x_\eta^2 + y_\eta^2$$

$$\beta = x_\xi x_\eta + y_\xi y_\eta$$

$$\gamma = x_\xi^2 + y_\xi^2 \tag{6}$$

are the coefficients obtained from the elliptic grid generation method. By knowing the values for a and b, Eq. (5) may be solved to obtain an expression for $T_{i,j}$. Eq. (5) is a quadratic one and an algebraic software such as Maple may be used to solve the equation in terms of $T_{i,j}$. The boundary condition equations also can be expanded and solved in a similar way. The direct heat conduction problem can be numerically solved to obtain the temperature distribution in the heat conducting body. By having the temperature values at any grid nodes as well as a and b, the thermal conductivity $k_T = a + bT$ can be calculated at any grid nodes (i,j).